Mathematical Reviews

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Vol. 20, No. 3 **MARCH, 1959** **Reviews 1621-2267**

LOGIC AND FOUNDATIONS

See also 1647.

1621:

***Dubarle, Le R. P. (H. D.) Initiation à la logique.** Collection de Logique Mathématique, Série A, XIII. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1957. 90 pp. 1400 francs.

It is welcomed that, among the recent not inconsiderable number of purely scientific books about mathematical logic, there is published now a book which successfully combines the philosophical and scientific aspects of this subject. It is written in a lucid philosophical style, and, one feels, it can be fully appreciated only by a reader interested in philosophy. It is intended for students of philosophy, who wish to obtain an idea of what is discussed in mathematical logic or must pass an examination on this subject. It does not contain detailed technicalities, and it has no exercises. It fulfills its aim in an admirable manner. It covers, in the first section, some of the main philosophical ideas of mathematical logic; in the second, scientific, section, elementary propositional logic, the basic functional calculus of the first order, the generalized functional calculus, and related theories; and, in the third section, the justification and limits of the logical-mathematical union.

B. Germansky (Jerusalem)

1622:

Glen, Oliver E. Monograph on cause and effect. Ann. Scuola Norm. Sup. Pisa (3) **11** (1957), 9–28.

1623:

v. Kempski, Jürgen. Relationen- und prädikatenlogische Untersuchungen zur Syllogistik. Arch. Math. Logik Grundlagenforsch. **2** (1956), 87–99.

1624:

Nolin, Louis. Algèbres de Boole et calcul des propositions. C. R. Acad. Sci. Paris **244** (1957), 1999–2002.

Démonstration purement algébrique du théorème de complétude du calcul des propositions classiques. L'auteur met en évidence le substrat algébrique des démonstrations connues. *Résumé de l'auteur*

1625:

***Nagel, Ernest; and Newman, James R. Gödel's proof.** New York University Press, New York, 1958. ix+118 pp. Paperbound \$1.75, clothbound \$2.95.

A detailed expository popular account.

1626:

Henkin, Leon. A generalization of the concept of ω-completeness. J. Symb. Logic **22** (1957), 1–14.

This is a counterpart to an earlier paper by the same author [A generalization of the concept of ω-consistency, same J. **19** (1954), 183–196; MR **16**, 103]. It deals with the following three properties of sets of sentences F in the first order predicate calculus with respect to a given set of individual constants, Γ. (i) If $B(x)$ is any formula defined in F and containing the single free variable x such that $B(a)$ is deducible from F for all a in Γ, then $(x)B(x)$ is deducible from F. (ii) There exists a function which assigns to every A in F and to every $B(x)$ as described, an individual constant of Γ, α_{AB}, such that, for any given A, the set of sentences $A \& (x)B(x) \supset B(\alpha_{AB})$ is consistent with F. (iii) For any sentence A, if A is defined in F and holds in all models of F whose individual constants are (denoted by) elements of Γ, then A is deducible from F. The three properties are called Γ-completeness, strong Γ-completeness, and Γ-saturation, respectively. (i) and (ii) are syntactical concepts while (iii) is semantic. The author shows that (iii) is equivalent to (ii) and entails (i). Also, (i) entails (iii) for finite or countable F. Two remarkable counter-examples show that this is not true for general F.

A. Robinson (Jerusalem)

1627:

Tarski, Alfred; and Vaught, Robert L. Arithmetical extensions of relational systems. Compositio Math. **13** (1958), 81–102.

The central notion of this paper is as follows. Let M be a relational system (or "structure") and let M' be an extension of M which contains the same relations. M' is said to be an 'arithmetical extension' of M if whenever a set of individual constants of M, $\{a_1, \cdots, a_n\}$, satisfies a formula $Q(x_1, \cdots, x_n)$ in M, where $n \geqq 0$, then $\{a_1, \cdots, a_n\}$ satisfies Q also in M'. This interesting notion is related to other concepts introduced recently into model-theory, such as Γ-completeness [L. Henkin; reviewed above] and model-completeness [A. Robinson, Complete theories, North-Holland Publishing Co., Amsterdam, 1956; MR **17**, 817]. The authors first establish two necessary and sufficient conditions for a structure M' to be an arithmetical extension of a structure M. Next they give two results which are related to the well-known theorem of Löwenheim and Skolem. These state that an infinite structure M of cardinal α which involves a finite number of relations only (i) is an arithmetical extension of structures of arbitrary infinite cardinal $\leqq \alpha$, and (ii) possesses proper arithmetical extensions of all powers $\geqq \alpha$. These results have to be qualified if the number of relations in M is infinite. With regard to (ii), this corrects a statement of the reviewer's [On the metamathematics of algebra, North-Holland Publ. Co., Amsterdam, 1951; MR **13**, 715; Th. 6.6.5] which requires the qualification that the number of relations in M does not exceed the number of individual constants. — Finally there are applications to various types of algebraic structures.

A. Robinson (Jerusalem)

1628:

***Klimovsky, Gregorio. Tres enunciados equivalentes al teorema de Zorn. [Three propositions equivalent to Zorn's theorem.]** Contribuciones Cientificas, Serie Matematica, Vol. 2, No. 1. Universidad de Buenos Aires, Facultad de Ciencias Exactas y Naturales, Buenos Aires, 1956. 29 pp.

The author shows that Zorn's lemma is equivalent to

the proposition that in a Boolean algebra every non-contradictory subset of an arbitrary set of elements is contained in a maximal non-contradictory subset. He proves also the corresponding result with "Boolean algebra" replaced by "general two-valued propositional calculus", a concept which he defines and describes. Finally he introduces the notion of a "simple syntactical system", and shows that Zorn's lemma is equivalent to the statement that every non-contradictory set of elements of a simple syntactical system is contained in a maximal non-contradictory set.

O. Frink (University Park, Pa.)

1629:

Rose, Alan. Sur les définitions de l'implication et de la négation dans certains systèmes de logique dont les valeurs forment des treillis. C. R. Acad. Sci. Paris **246** (1958), 2091–2094.

Anstatt 2-stellige Verknüpfungen über der Menge $\{0, 1\}$ zu definieren — sodass die "Polynome" die identisch $=1$ sind, gerade die "wahren" Formeln der klassischen Logik liefern — kann man z.B. $\{0, 1\}$ durch einen endlichen Verband ersetzen. Leider spricht man auch dann noch von "Aussagenlogik". Verf. behandelt im Anschluss an Birkhoff [Lattice theory, Amer. Math. Soc., New York, 1948; MR **10**, 673; p. 197] Verbände aus gewissen Paaren (x, j), wobei x rational, j ganz ist. Die klassische Implikation wird ersetzt durch

$$(x, j)\to(y, k)=\begin{cases}(\min(1, 1-x+y), j) & \text{für } j=k,\\ (0, j)=(0, k) & \text{für } j\neq k,\end{cases}$$

und es werden einige Polynome angegeben, die identisch$=(1, j)=(1, k)$ sind. Verf. selbst beurteilt seine Definition als nicht "entièrement satisfaisante", stellt aber eine weitere Arbeit über die Formalisierung seiner "Logik" in Aussicht.

P. Lorenzen (Kiel)

SET THEORY

See also 1637, 1649.

1630:

Erdös, P.; and Hajnal, A. On the structure of set-mappings. Acta Math. Acad. Sci. Hungar. **9** (1958), 111–131.

The symbol $(m, n, t)\to p$ stands for the following proposition. Let S be a set of power m. Let f be a mapping defined on the set of all subsets of S of cardinal t, such that $f(X)\subseteq S$, $f(X)\cap X=\emptyset$, and $|f(X)|<n$. Then S has a subset S' of power p such that $f(X)\cap S'=\emptyset$ for all $X\subseteq S'$ (and $|X|=t$).

The symbol $(m, n, \omega)\to p$ stands for the corresponding proposition for a mapping defined on the set of all finite subsets of S. The negations are indicated by $\nrightarrow$. [For background, see P. Erdös, Proc. Amer. Math. Soc. **1** (1950), 127–141; MR **12**, 14; and Erdös and Fodor, Acta. Sci. Math. Szeged. **18** (1957), 243–260; MR **19**, 1152.]

Typical results for the case in which m is infinite: (A) If t is infinite, then $(m, 2, t)\nrightarrow t$. (B) If $m<\aleph_\omega$, then $(m, 2, \omega)\nrightarrow\aleph_0$. (C) Under the generalized continuum hypothesis (g.c.h.), $(\aleph_{\alpha+k}, \aleph_\alpha, k)\to\aleph_{\alpha+1}$ for finite k. (D) Under the g.c.h. $(m, n, k)\to m$ if m is singular, $n<m$, and k is finite. (E) If m is strongly inaccessible, and if a set of this power admits a Ulam measure, then $(m, n, \omega)\to m$ for $n<m$.

Open problems: (1) $(\aleph_\omega, 2, \omega)\to\aleph_0$? (note: $\nrightarrow\aleph_1$). (2) $(\aleph_2, \aleph_0, 3)\to\aleph_2$? (note: $\to\aleph_1$ but $\nrightarrow\aleph_2$). (3) $(\aleph_2, 2, 3)\to\aleph_1$? (note: $\to\aleph_0$ but $\nrightarrow\aleph_2$).

L. Gillman (Princeton, N.J.)

1631:

Neumer, Walter. Algorithmen für Ordnungszahlen und Normalfunktionen. I. Z. Math. Logik Grundlagen Math. **3** (1957), 108–150.

First the free algorithm for arbitrary ordinal numbers, introduced by the author [Math. Z. **59** (1954), 434–454; MR **15**, 689], is described here from a more general standpoint. Then a "facient algorithm" is developed which initially derives, from a distinguished sequence of operators, stronger operators by means of certain limit operations. There results, in particular, an extended free algorithm for ordinal numbers, corresponding to the author's extended constructive algorithm [ibid. **64** (1956), 435–456; MR **18**, 139].

F. Bagemihl (Notre Dame, Ind.)

COMBINATORIAL ANALYSIS

1632:

†Netto, Eugen. Lehrbuch der Combinatorik. Zweite Auflage, erweitert und mit Anmerkungen versehen von V. Brun und Th. Skolem. Chelsea Publishing Company, New York, 1958. viii+341 pp. $4.95.

A reproduction of the second edition [Teubner, Leipzig, 1927]. The first edition of this well-known work appeared in 1901 [ibid.].

1633:

Wyman, Max; and Moser, Leo. On the problème des ménages. Canad. J. Math. **10** (1958), 468–480.

The numbers, U_n, of the reduced problème des ménages (the number of permutations on n elements discordant with two permutations, one of which is a circular permutation of the other) are given a new representation in terms of Bessel functions and integrals of Bessel functions. This is used to show that the asymptotic formula of Kaplansky and Riordan [Scripta Math. **12** (1946), 113–124; MR **8**, 365], namely $U_n\sim ne^{-2}\sum(-1)^k(n-1-k)!/k!$, is an exact formula when limited to its finite part and rounded off to the nearest integer. Another expression, $U_n=2T_n(k/2)$ with k the Blissard variable: $k^n\equiv k_n=(E-2)^n 0!$, $(E^n 0!=n!)$, is given {this has been anticipated by Yamamoto [Mem. Fac. Sci. Kyusyu Univ. Ser. A **10** (1956), 1–13; MR **17**, 1174]}. A table of the numbers, computed by F. L. Miksa, for all $n\leq 65$ is presented.

J. Riordan (New York, N.Y.)

1634:

Mendelsohn, N. S.; and Dulmage, A. L. The convex hull of sub-permutation matrices. Proc. Amer. Math. Soc. **9** (1958), 253–254.

Let $A=[a_{ij}]$ be an n by n matrix with non-negative real entries. Let S represent the sum of all entries of A. Let R_i denote the sum of row i of A and let C_j denote the sum of column j of A. Define $M=\max(R_i, C_j)$. A matrix P is a "sub-permutation matrix of rank" r provided each entry of P is 1 or 0; each row and each column of P contains at most one 1; P contains exactly r 1's. The authors prove that a necessary and sufficient condition that A lie in the convex hull of all sub-permutation matrices of rank $n-i$ is that $S=n-i$ and $(n-i)/n\leq M\leq 1$. For $M=1$ and $S=n$ this reduces to the well known result that the convex hull of the set of all n by n permutation matrices is the set of all n by n doubly stochastic matrices.

H. J. Ryser (Columbus, Ohio)

1635:

Mendelsohn, N. S.; and Dulmage, A. L. Some generalizations of the problem of distinct representatives. Canad. J. Math. **10** (1958), 230–241.

Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_m$ $(n \leqq m)$ be two sets of elements. A set S of incidences $(a_{q_1}, b_{r_1}), \cdots, (a_{q_s}, b_{r_s})$ defined relative to a given dyadic relation is "regular" provided no a or b appears more than once among the components. A necessary and sufficient condition for the existence of a regular S in which $a_1, \cdots, a_r$ and $b_1, \cdots, b_s$ appear is the following. (1) For $k=1, \cdots, r$, every subset of k of the elements of $a_1, \cdots, a_r$ are incident with at least k distinct elements of $b_1, \cdots, b_m$. (2) For $p=1, \cdots, s$, every subset of p of the elements of $b_1, \cdots, b_s$ are incident with at least p distinct elements of $a_1, \cdots, a_n$. This result may be viewed as an extension of the Hoffman-Kuhn theorem concerning systems of distinct representatives.

Now let A, S, M, and sub-permutation matrix of rank r be defined as in the preceding review. The set of places at which the 1's appear in a sub-permutation matrix of rank r is a "sub-permutation set of places of rank" r. If $(n-1)M<S$, then A has non-zero entries in a permutation set of places. If $1/(n-1) \leqq M/S < (n-1)/(n^2-2n)$, then A has non-zero entries in a sub-permutation set of places of rank $(n-1)$. To handle the general case numbers $P_i(n)$ are introduced satisfying

$$P_i(n)=n^i-\binom{i}{1}n^{i-1}+\binom{i-1}{2}n^{i-2}-\cdots.$$

The main result asserts that if $r>2$, $n \geqq r$, and

$$(*) \qquad \frac{P_{r-2}(n)}{P_{r-1}(n)} \leqq \frac{M}{S} < \frac{P_{r-1}(n)}{P_r(n)},$$

then A has non-zero entries in a sub-permutation set of places of rank $n-r+1$. The authors have conjectured that (*) may be replaced by the simpler condition

$$\frac{1}{n-r+1} \leqq \frac{M}{S} < \frac{1}{n-r},$$

which would be the best possible. A note added in proof asserts that this conjecture has been established. Its proof along with results involving term rank and a new concept called stochastic rank are to appear.

H. J. Ryser (Columbus, Ohio)

1636:

Miksa, F. L.; Moser, L.; and Wyman, M. Restricted partitions of finite sets. Canad. Math. Bull. **1** (1958), 87–96.

If $G_{n,r}$ denotes the number of ways in which n distinguishable objects can be placed in an unrestricted number of indistinguishable boxes when each box can hold at most r objects, then the authors of this paper prove among other results the following:

$$\sum_{n=0}^{\infty} G_{n,r}\frac{x^n}{n!}=\exp\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^r}{r!}\right);$$

if p be a prime $>r$, $G_{n+p,r} \equiv G_{n,r} \pmod{p}$.

An asymptotic formula for $G_{n,r}$ when r is fixed and n large is given. Polynomials $G_{n,r}(t)$ defined by the relation

$$\sum_{n=0}^{\infty} G_{n,r}(t)\frac{x^n}{n!}=\exp\left(t(x+\frac{x^2}{2!}+\cdots+\frac{x^r}{r!})\right)$$

are also considered. *H. Gupta* (Chandigarh)

ORDER

See also 1624, 2255

1637:

Gillman, Leonard. A continuous exact set. Proc. Amer. Math. Soc. **9** (1958), 412–418.

Inspired by a question of Cuesta [Revista Mat. Hisp.-Amer. (4) **5** (1945), 191–203; MR **7**, 277], the author defines a continuous ordered set of power $2^{\aleph_0}$ which is shown, under the hypothesis that $2^{\aleph_0}$ is regular, to be similar to no proper subset of itself.

F. Bagemihl (Notre Dame, Ind.)

1638:

Hiraguchi, Tosio. On the λ-dimension of the product of orders. Sci. Rep. Kanazawa Univ. **5** (1956), 1–5.

For a partially ordered set P, denote by $D_\lambda[P]$ the smallest cardinal S such that P is a subset of a cardinal product of S copies of the real number system: If A and T are denumerably infinite and L is a linear order on A, then it is shown that $D_\lambda[L^T]=\aleph_0$ and $D_\lambda[\lambda^T]=\aleph_0$. An upper bound for $D_\lambda[P]$ is obtained if P is the cardinal product $\prod_T P_t$ of denumerably many orderings P_t for each of which D_λ exists, in terms of the dimensions of mappings of T into the realizers of the P_t. [Cf. Hiraguchi, same Rep. **2** (1953), 1–3; **4** (1955), 1–20; MR **17**, 937, 1045; and S. Ginsburg, Amer. J. Math. **76** (1954), 590–598; MR **15**, 943.] *P. M. Whitman* (Baltimore, Md.)

1639:

Kosmák, L.; Koutský, K.; und Novotný, M. Über additiv irreduzible Elemente und additive Basen im Verbande. Publ. Fac. Sci. Univ. Masaryk **1956**, 165–175. (Czech and Russian summaries)

The authors introduce the notions of an (additively) m-irreducible element of a lattice, and of an (additive) m-basis, where m is any cardinal number, and study relations between these concepts. An element x of a lattice is said to be m-irreducible if it is not the join of a set, of cardinal number less than m, of lattice elements all distinct from x. A set B of lattice elements is said to be an m-basis if every lattice element is the join of a non-empty set of elements of B of cardinal number less than m.

It is shown that an element is m-irreducible if and only if it is a member of every m-basis. The union, join, and meet of two m-bases is an m-basis, provided m is a regular cardinal number. If m is infinite, it is shown that a lattice has a unique m-basis if and only if every lattice element is m-irreducible, and has a smallest m-basis if and only if the set of all m-irreducible elements is an m-basis. The authors characterize lattices possessing an m-basis in terms of which the representation of every lattice element is unique. *O. Frink* (University Park, Pa.)

1640:

Matsushita, Shin-ichi. Ideals in non-commutative lattices. Proc. Japan Acad. **34** (1958), 407–410.

An algebraic system A, having two associative and idempotent (but not necessarily commutative) operations $*$ and $\cdot$, is given; conditions for the existence of an order relation on A are discussed. An (l, l)-ideal is a subsystem M of A such that if $a \in M$ and $p \in A$ then $a*p \in M$ and $a \cdot p \in M$ {this differs from the usual notion}. In suitable cases, minimal ideals exist and systems can be decomposed into sums thereof. This repeats and somewhat amplifies the author's earlier publication [C. R. Acad. Sci. Paris **236** (1953), 1525–1527; MR **14**, 838].

P. M. Whitman (Baltimore, Md.)

1641:

Kalman, J. A. Lattices with involution. Trans. Amer. Math. Soc. **87** (1958), 485–491.

The author calls a lattice an i-lattice if it has an involution $x \to x'$ (anti-automorphism of the order 2). A distributive i-lattice L for which $x \cap x' \leqq y \cup y'$ for x and $y \in L$ is called a normal i-lattice. Boolean algebras and l-groups are normal i-lattices under complementation and inverse operation, respectively. An element z of an i-lattice is called a zero if $z' = z$. It is proved that every distributive i-lattice is i-isomorphic with an i-sublattice of a direct union of replicas of the i-lattice with four elements and two zeros. Furthermore, every normal i-lattice is i-isomorphic with an i-sublattice of a vector lattice. Several types of i-lattices are classified with respect to certain laws. If L is an i-lattice with a unique zero 0, then an element $x \in L$ is called positive if $x \geqq 0$. The author determines all the normal i-lattices which have a given sublattice of positive elements. Finally some imbedding theorems are proved.

Ph. Dwinger (Lafayette, Ind.)

1642:

Benado, Mihail. Über die allgemeine Theorie der regulären Produkte von Herrn O. N. Golowin. II. Math. Nachr. **16** (1957), 137–194.

The lattice theoretic content of some results on group products is abstracted from part I of the same title [same Nachr. **14** (1955), 213–234 (1956); MR **18**, 871], the results having been announced in an earlier note [Acad. R. P. Romîne. Bull. Şti. Şect. Şti. Mat. Fiz. **7** (1955), 249–254; MR **17**, 1050]. On a complete lattice S with 0 and 1, a relation N is to hold for some of the pairs a, b where $a \geqq b$. Whenever aNb, one requires that (1) $a \geqq x \geqq y$ implies $x \wedge (b \vee y) = (x \wedge b) \vee y$; (2) $a \geqq y \geqq b$, $a \geqq x$ imply $y \wedge (x \vee b) = (y \wedge x) \vee b$; (3) $a \geqq x$ implies $xN(x \wedge b)$; (4) $a \geqq x$, xNy imply $(b \vee x)N(b \vee y)$. Further, (5) $1N1$, $1N0$; and (6) $S_N = E[a|\,1Na]$ is a complete modular sublattice of S. Define a closure operation on S by $\bar{a} = \bigwedge b$, the intersection taken over all $b \geqq a$, $b \in S_N$. If 1 is a union of elements a, each disjoint from the closure of the union of the remaining a's, 1 is said to be a regular product of the a's, an abstraction from the concept of regular product of groups in the sense of Golovin in part I. The author shows that the regular products are 1-1 correspondence with certain (Fitting) systems of endomorphisms of S (order-preserving maps). Isomorphism of two regular products is defined and characterized. Necessary and sufficient conditions are given for the refinements of two regular products to have the closures of the elements of the two refinements equal, two by two, and to have the elements of the two refinements carried onto each other by suitable Fitting endomorphisms associated with these products. An example of such a lattice S with a relation N is given by the lattice of subrings of a ring R, where ANB means that A and B are subrings of R and that B is a two-sided ideal in A. The sublattice S_N is just the set of two-sided ideals in R.

F. Haimo (St. Louis, Mo.)

1643:

Benado, Mihail. Sur une interprétation topologique de la notion de normalité unitaire. Bull. Sci. Math. (2) **81** (1957), 87–112.

Let $x \to \bar{x}$ be a topological closure operator on the lattice S of all subsets of a set. From this closure operation, one obtains a relative closure by defining, for $x \geqq y$, $\varphi(x, y) = x \cap \bar{y}$. The following conditions are easily verified: FR1. $\varphi(x, x) = x$ for all $x \in S$; FR2. $\varphi(x, \varphi(x, y)) \leqq \varphi(x, y)$ if $x \geqq y$; FR3. $x \geqq y \geqq z$ implies $\varphi(x, y) \geqq \varphi(x, z) \geqq \varphi(y, z)$; FR4. $\varphi(z \cup x, z \cup y) \leqq \varphi(z \cup x, z) \cup \varphi(x, y)$ for $x \geqq y$ and $z \in S$. Associated with the relative closure operator φ is the binary relation "y is relatively closed in x": $y = \varphi(x, y)$ (or, in other words, $x \cap \bar{y} = y$) for $x \geqq y$. Denote this relationship xKy. This relation satisfies Y1. xKy, $x \geqq z$ implies $zK(y \cap z)$; Y2. xKy, $x \geqq z$ and zKw together imply $(y \cup z)K(y \cup w)$; Y3. if $x \geqq y$, then the set $\{z \in S|\ xKz, z \geqq y\}$, if it is non-empty, has a smallest element; Y4. xKx for all $x \in S$. The paper under review presents an axiomatic study of relative closure operations satisfying FR1–4 (or related conditions) and associated binary relations satisfying Y1–4 (or variations of these), all defined on general lattices. The motive for this study comes from the observation that such relations and closure operations occur quite naturally. For example, if S is a complete modular lattice with the descending chain condition, the relation K, defined by xKy if $x \geqq y$ and $y \cup (z \cup W) = (y \cap z) \cup (y \cap w)$ for all $z, w \leqq x$, satisfies Y1–4. The following theorem is a typical sample of the results contained in the paper: if S is any lattice, there is a one-to-one correspondence between the relative closure operations $\varphi(x, y)$ satisfying FR1–4 and the binary relations xKy (defined for $x \geqq y$) satisfying Y1–4.

R. S. Pierce (Seattle, Wash.)

1644:

Curzio, Mario. Su di una questione proposta da G. Birkhoff. Ricerche Mat. **6** (1957), 27–33.

The author extends his study of modular lattices of length three to supermodular lattices of this length. He shows that such lattices, if non-complemented, non-modular and of order $n \geqq 7$, contain sublattices of every order $m < n$.

G. Birkhoff (Cambridge, Mass.)

1645:

Wolk, E. S. A characterization of distributive lattices. Tôhoku Math. J. (2) **10** (1958), 32–36.

Removing the assumption of the existence of lattice operations in a criterion of Iseki [Acta Math. Acad. Hungar. **3** (1952), 241–242; MR **14**, 717], the author shows that any partially ordered set P is a distributive lattice if and only if there exists an inductive dual imbedding operator ψ on P such that the union of each set of completely irreducible ψ-ideals is prime and the union of their complements is coprime [for terminology see, e.g., Dilworth and McLaughlin, Duke Math. J. **19** (1952), 683–693; MR **14**, 717; "dual" means that single elements are mapped into dual principal ideals]. Thence a Boolean algebra is characterized in terms of order properties, by the existence of generalized complements and the behavior of ideals. The basic tool is the use of ψ to generalize Birkhoff and Frink's theorem [Trans. Amer. Math. Soc. **64** (1948), 299–316; MR **10**, 279] on representing lattices by sets.

P. M. Whitman (Baltimore, Md.)

1646:

Ginsburg, Seymour. On the existence of complete Boolean algebras whose principal ideals are isomorphic to each other. Proc. Amer. Math. Soc. **9** (1958), 130–132.

It is shown that for each infinite cardinal $\aleph_\alpha$ there exists a complete Boolean algebra of power $2^{\aleph_\alpha}$, with the property that all of its non-zero principal ideals are isomorphic. In general, this algebra is not unique up to isomorphism.

B. Jónsson (Minneapolis, Minn.)

GENERAL ALGEBRAIC SYSTEMS

See also 1627.

1647:

Mal'cev, A. I. On derived operations and predicates. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 24–27. (Russian)

Let A be an algebra with fundamental operations $f_i(x_1, \cdots, x_{m_i})$ $(i=1, \cdots, s)$. $P_i(x_1, \cdots, x_{m_i}, y)$ is the predicate $f_i(x_1, \cdots, x_{m_i})=y$. Let $\mathfrak{A}$ be a formula of the lower predicate calculus LPC, built up from the P_i and containing the free variables $x_1, \cdots, x_m, y$. $\Phi(\mathfrak{A})$ is the formula expressing that $\mathfrak{A}$ defines y as a function of $x_1, \cdots, x_m$. A formula is called "universal" if its prenex form contains only generalizing quantifiers. Th. 1: Let K be the conjunction of a finite or infinite number of closed universal formulas, built up from the P_i, $\mathfrak{A}$ a formula as above. Suppose that (1) $\bigwedge_i \Phi(P_i) \wedge K \to \Phi(\mathfrak{A})$ is a theorem of LPC. Then formulas $\mathfrak{A}_j$ and terms g_j can be found, such that (2) $\bigvee_j \mathfrak{A}_j$, (3) $\mathfrak{A}_p \wedge \mathfrak{A}_q \leftrightarrow g_i=g_j$ and (4) $\mathfrak{A} \leftrightarrow \bigwedge_j (\mathfrak{A}_j \leftrightarrow g_j=y)$ are consequences of the system of axioms $\mathfrak{S}=\{\Phi(P_1), \cdots, \Phi(P_s), K\}$. Theorem 2: Let K^* be a class of models in which predicates P_i, P are defined. The following properties (I) and (II) are equivalent. (I) For some finite or infinite set of open formulas $\mathfrak{B}_\alpha$,

$$P(x_1, \cdots, x_n) \leftrightarrow \bigwedge_\alpha (x_{n+1}) \cdots (x_p) \mathfrak{B}_\alpha(x_1, \cdots, x_n, x_{n+1}, \cdots, x_p)$$

is true in every model in K^*. (II) If σ is a $(P_1, \cdots, P_s)$-isomorphism between a model M, containing $P_1 \cdots P_s$ and a partial predicate P, and a K^*-model M^*, while every $(P_1 \cdots P_s, P)$-submodel of the partial model M can be imbedded in some K^*-model, then σ is also a P-homomorphism from M to M^*. Theorem 2 contains as special cases theorems of Tarski [Nederl. Akad. Wetensch. Proc. Ser. A. **57** (1954), 582–588, Th. 1.12; MR **16**, 554] and of A. Robinson [J. Symb. Logic **21** (1956), 33–35; MR **17**, 817].

Some special results are obtained for quasi-primitive classes of algebras. *A. Heyting* (Amsterdam)

1648:

Schauffler, Rudolf. Die Assoziativität im Ganzen, besonders bei Quasigruppen. Math. Z. **67** (1957), 428–435.

Let L be a non-empty set of internal laws of composition on a finite set S. The author defines and discusses simple properties of L and gives examples. In particular he calls L associative on the whole if to every couple τ_1, τ_2 of composition laws of L there exist τ_3, τ_4, τ_5, $\tau_6 \in L$ such that $(x\tau_1 y)\tau_2 z = x\tau_3(y\tau_4 z)$ and $x\tau_1(y\tau_2 z)=(x\tau_5 y)\tau_6 z$ hold for all $x, y, z \in S$. The set of all composition laws on S which make S a quasigroup is then shown to be associative on the whole if and only if S has at most three elements. *A. Jaeger* (Cincinnati, Ohio)

1649:

Opluštil, Karel. Die O-Systeme. Publ. Fac. Sci. Univ. Masaryk **1957**, 69–86. (Russian summary)

An O-system M is a set of elements in which addition and multiplication are defined and satisfy the following postulates: (1) addition is associative; (2) if $a \neq b$ exactly one of the equations $a+x=b$, $b+y=a$ is solvable in M, if $a=b$ then both are solvable in M; (3) multiplication is associative; (4) $a(b+c)=ab+ac$. Postulate (2) leads at once to an order relation in M. Study of this particular set of postulates is motivated by the fact that they are nontrivially satisfied by the set of all ordinal numbers of countable sets. The author derives for O-systems the usual algebraic theorems concerned with subsystems, homomorphisms, zero divisors, units, etc.

D. C. Murdoch (Vancouver, B.C.)

THEORY OF NUMBERS

See also 1679, 1865.

1650:

Mitrinovitch, Dragoslav S. Sur quelques formules sommatoires. Univ. Beogradu. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. no. **7** (1956), 8 pp. (French. Serbo-Croatian summary)

The following formula is proved, among others:

$$\sum_{k=0}^{m} \left\{(-1)^k \binom{m}{k} \Big/ \prod_{i=1}^{\nu} [n+(i+k-1)p]\right\} \equiv (\nu+m-1)!/[p^\nu(\nu-1)!(n/p)_{\nu+m}],$$

where m, ν are natural numbers, p and n are positive reals, and

$$(n/p)_{\nu+m} \equiv \prod_{s=1}^{\nu+m} [(n/p)+s-1].$$

Several interesting corollaries are given.

1651:

Srinivasan, M. S. The enumeration of positive rational numbers. Proc. Indian Acad. Sci. Sect. A. **47** (1958), 12–24.

By finding the order of the median series in which a given rational number first occurs, and the position in the series in which it occurs, a one-to-one correspondence is set up between the positive integers and the rational numbers greater than 1. The technique used brings in the Euclidean algorithm and associated continued fractions. *H. A. Thurston* (Vancouver, B.C.)

1652:

Parker, R. V. Sums of powers of the integers. Math. Gaz. **42** (1958), 91–95.

Let $S_n(x)=\sum_{a=1}^{x} a^n$. By taking the first differences of the function $x^n(x+1)^n$ the author proves that, for n odd, $S_n(x)$ can be expressed as a polynomial in $x(x+1)$. In a similar way he proves that, for n even, $S_n(x)$ can be expressed as such a polynomial multiplied by $2x+1$. These results had been derived by Piza by a somewhat different method (Math. Mag. **25** (1952), 137–142; MR **13**, 625). The relation of the resulting formulas to difference tables is also mentioned. *H. W. Brinkmann* (Swarthmore, Pa.)

1653:

Ząbek, Światomir. Sur la périodicité modulo m des suites de nombres $\binom{n}{k}$. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **10** (1956), 37–47 (1958). (Polish and Russian summaries)

Several lemmas and theorems lead up to the main theorem (all letters denoting positive integers): The length q_k of the shortest period modulo $m=p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ of the sequence $S=\left\{\binom{n}{k}\right\}$ $(n=1, 2, 3, \cdots)$, k fixed, is given by $q_k=m \cdot p_1^{\beta_1} \cdots p_t^{\beta_t}$, where each β_i satisfies $p_i^{\beta_i} \leqq k < p_i^{\beta_i+1}$.

Really, the arguments used suffice to prove a more general theorem which is obtained by replacing the

sequence S by a sequence $\{c_n\}$, where the integers c_n, positive, negative or zero, are the elements of an arithmetical progression of order k, with the constant kth difference $=1$ (or, at least, relatively prime to m).

A priority claim made for W. Sierpiński as the first to investigate sequences periodic modulo m is not correct.

A. J. Kempner (Boulder, Colo.)

1654:

Erdős, Paul. On an elementary problem in number theory. Canad. Math. Bull. **1** (1958), 5–8.

Quoting from the paper: "Let $0<x\leq y$. Estimate the smallest $f(x)$ so that there should exist integers u and v satisfying (1) $0\leq u, v<f(x)$ and $(x+u, y+v)=1$. — I am going to prove that for every $\varepsilon>0$ there exist arbitrarily large values of x satisfying (2) $f(x)>(1-\varepsilon)(\log x/\log\log x)^{\frac{1}{2}}$ but that for a certain $c>0$ and all x, and (3) $f(x)<c\log x/\log\log x$. — A sharp estimation of $f(x)$ seems to be a difficult problem. It is clear that $f(p)=2$ for all primes p. ... By similar methods as used in the proof of (3) we can prove the following theorem. Let $g(x)(\log x/\log\log x)^{-1}\to\infty$, $0<x<y$. Then the number of pairs $0\leq u, v<g(x)$ satisfying $(x+u, y+v)=1$ equals $(1+o(1))(6/\pi^2)g^2(x)$. ... We can show by methods used in the proof of (2) in our theorem that we cannot have $g(x)$ less than $c(\log x/\log\log x)^{\frac{1}{2}}$, i.e., $g(x)(\log x/\log\log x)^{-\frac{1}{2}}\to\infty$ is necessary for the truth of our theorem. An exact estimation of $g(x)$seems difficult."

A footnote states: "L. Moser informs me that he independently obtained this result (2) and its generalization to an m-dimensional lattice."

A. J. Kempner (Boulder, Colo.)

1655:

Schinzel, A.; and Wang, Y. A note on some properties of the functions $\varphi(n)$, $\sigma(n)$ and $\theta(n)$. Ann. Polon. Math. **4** (1958), 201–213.

Let ε and $a_1, \cdots, a_h$ be fixed positive numbers. It is shown, using a modified Brun's method, that there are positive numbers c and X_0, depending on ε and the a's, such that for $X>X_0$, the number of positive integers $n\leq X$, for which

$$(*)\qquad \left|\frac{\varphi(n+i)}{\varphi(n+i-1)}-a_i\right|<\varepsilon \quad (i=1, 2, \cdots, h)$$

is greater than $cX/\log^{h+1}X$. This refines an earlier theorem of the first author [Bull. Acad. Polon. Sci. Cl. III **3** (1955), 415–419; MR **17**, 461] which asserted that the number of solutions of (*) is infinite. The theorem remains true if $\varphi(n)$ is replaced by $\sigma(n)$. A somewhat weaker result holds for $\theta(n)$ (the number of divisors of n), which implies that for arbitrary positive $a_1, \cdots, a_h$ there is a sequence $n_1, n_2, \cdots$ of positive integers such that

$$\lim_{k\to\infty}\frac{\theta(n_k+i)}{\theta(n_k+i-1)}=a_i \quad (i=1, 2, \cdots, h).$$

W. J. LeVeque (Göttingen)

1656:

Barreneche, Rodrigo Noguera. Solution of an equation related to Fermat's last theorem. Stvdia. Rev. Univ. Atlantico **2** (1957), 91–101. (Spanish)

1657:

Barreneche, Rodrigo Noguera. Elementary discussion of Fermat's last theorem. Stvdia. Rev. Univ. Atlantico **2** (1957), 103–198. (Spanish)

1658:

Barreneche, Rodrigo Noguera. Historically the first proof incontrovertible, complete and universal, of the grand theorem of Fermat with the Davidic algebra of the "principle of the amateurs" in mathematical investigation. Stvdia. Rev. Univ. Atlantico **2** (1957), 199–209. (Spanish)

1659:

Barreneche, Rodrigo Noguera. The problem of rational quadrature of the right triangle with integral sides and other minutiae. Stvdia. Rev. Univ. Atlantico **2** (1957), 211–221. (Spanish)

1660:

Zeckendorf, E. Équations quadratiques à discriminant atypique. Bull. Soc. Roy. Sci. Liège **27** (1958), 28–40.

In an earlier paper [same Bull. **26** (1957), 112–122; MR **19**, 730] the author studied equations of the type $A^2-MB^2=k_1(-m)^x$, $M=v^2+m$, and showed how to express solutions in integers A and B as terms of certain recurring series. Now the more general case $M\neq v^2+m$ is considered.

W. Ljunggren (Blindern)

1661:

Buzby, B.; and Whaples, G. Quadratic forms over arbitrary fields. Proc. Amer. Math. Soc. **9** (1958), 335–339; erratum, **10** (1959), 174.

The authors give another proof of a well-known theorem of E. Witt [J. Reine Angew. Math. **176** (1936), 31–44] to the effect that two binary or ternary quadratic forms over an arbitrary field (of characteristic not two) are equivalent if and only if they have the same determinant and Hasse invariant. The main idea of simplifying the proof is to replace the theory of simple algebras by the cohomology theory of finite groups.

Y. Kawada (Tokyo)

1662:

Malyshev, A. V. The distribution of integer points on a four-dimensional sphere. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 25–28. (Russian)

The following theorem is stated, and a proof of it is lightly sketched: Let Ω be a 4-dimensional convex conical region with vertex at the origin and with vertex solid angle $\omega>0$. Let n be an odd positive integer, and let $r(\omega, n)$ be the number of integral points on the sphere $x_1^2+x_2^2+x_3^2+x_4^2=n$ and lying in Ω. Then for fixed ω,

$$r(\omega, n)=(2\pi^2)^{-1}\omega r(n)(1+O(n^{-1/18+\epsilon})) \text{ as } n\to\infty,$$

where $r(n)$ is the total number of integral points on the sphere. Similar results are stated concerning the number of primitive points in Ω, and the number of points in Ω whose coordinates lie in specified residue classes.

W. J. LeVeque (Göttingen)

1663:

Kosmák, Ladislav. Bemerkung über die nichtnegativen ganzzahligen Lösungen der Gleichung $\sum_{i=1}^{k} r_i=n$. Časopis Pěst. Mat. **83** (1958), 80–82. (Czech. Russian and German summaries)

The author shows how the proof of a certain theorem of K. Čulik [Časopis Pěst. Mat. **82** (1957), 353–359; MR **19**, 531] may be simplified; thereby he arrives at a slightly sharper form of this theorem, which now reads: If $\sum_{i=1}^{k} h_i=n$ is a partition of n into non-negative integers which satisfy also the condition $\max_{1\leq i\leq k} h_i-\min_{1\leq i\leq k} h_i\leq 1$, if $\sum_{i=1}^{k} r_i$ is any partition of n into non-negative integers, and if c is a given positive integer, then

$$\sum_{i=1}^{k}\binom{h_i}{c}<\sum_{i=1}^{k}\binom{r_i}{c}$$

if, and only if, $2\leq c\leq \max_{1\leq i\leq k} r_i > 1+\min_{1\leq i\leq k} r_i$; otherwise the two expressions are equal.

H. Halberstam (London)

1664:

Schinzel, A.; et Sierpiński, W. Sur les sommes de quatre cubes. Acta Arith. 4 (1958), 20–30.

The second author has conjectured that every integer g can be represented in infinitely many ways in the form $g=x^3+y^3-z^3-t^3$, where x, y, z, t are positive integers. This conjecture is here confirmed for all except integers $g=hn^3$ with $h\equiv\pm 2, \pm 4$, or $\pm 5 \pmod{18}$, n arbitrary. The result is obtained by first proving the conjecture for any integer g having a representation, $g=a^3+b^3-c^3-d^3$, where a, b, c, d are any integers such that $(a+b)(c+d)$ is a positive non-square and $a\neq b$ or $c\neq d$. By applying this theorem in the cases not already covered by the above result, the authors verify the conjecture for $|g|\leq 100$; and this has been extended by A. Makowski to $|g|\leq 300$ except for $g=148$, 257, and 284. They also obtain as an easy consequence of their main result the theorem that every positive rational has infinitely many representations in the form, $x^3+y^3-z^3-t^3$, with x, y, z, t positive rationals. They call attention to an error in the proof by C. A. Mebius [Göteborgs Kungl. Vetenskaps- och Vitterhets-Samhälles Handlingar (6) Ser. B 3, no. 6, (1945); MR 8, 6] that any integer expressible as the (algebraic) sum of 4 integral cubes can be so expressed in infinitely many ways.

R. J. Levit (San Francisco, Calif.)

1665:

Kurbatov, V. A. On equations of prime degree. Mat. Sb. N.S. **43(85)** (1957), 349–366. (Russian)

Let Ω be a number field (the results are applicable with suitable modifications to arbitrary fields of characteristic 0) and $f(x)$ an irreducible polynomial with coefficients in Ω, of prime degree n and with the roots $x_0, x_1, \cdots, x_{n-1}$. Let ε be a primitive nth root of unity and $\theta(x)$ a polynomial with coefficients in $\Omega(\varepsilon)$ for which the values $\theta_k=\theta(x_k)$ form a complete class of conjugates with respect to $\Omega(\varepsilon)$. Suppose now that the θ_k do not all belong to $\Omega(\varepsilon)$, but that they are linearly dependent over $\Omega(\varepsilon)$ with coefficients that are not all equal. We form the sums $S(l)=n^{-1}\sum_{k=0}^{n-1}\varepsilon^{kl}\theta(x_k)$ $(l=0, 1, \cdots, n-1)$. Then the author proves that among the $S(l)$ with $l\neq 0$ there are some that are zero and some that are not. Those that are zero form a basis of the space of linear relations among the θ_k over $\Omega(\varepsilon)$; those that are not zero are linearly independent over $\Omega(\varepsilon)$. This theorem leads to the result that the equation $f(x)=0$ is soluble by radicals if in the above notation some conjugates θ_k, not contained in $\Omega(\varepsilon)$, are connected by a linear relation with coefficients in $\Omega(\varepsilon)$ that are not all equal. This in turn yields the criterion: an equation of prime degree is soluble by radicals if and only if its roots generate a linearly dependent class of conjugate elements relative to some field $\Omega(\varepsilon)$. In this formulation ω can be taken as the minimal field containing the coefficients of the equation or one that is obtained from it by adjoining the square root of the discriminant, the equation is assumed to be irreducible, the conjugate elements do not belong to $\Omega(\varepsilon)$ and the linear dependence, as before, is not symmetrical.

K. A. Hirsch (London)

1666:

Rogers, K.; and Swinnerton-Dyer, H. P. F. The geometry of numbers over algebraic number fields. Trans. Amer. Math. Soc. **88** (1958), 227–242.

The authors extend Mahler's compactness theorem on ordinary lattices [Proc. Royal Soc. London, Ser. A **187** (1946), 151–187; C. Chabauty, Bull. Soc. Math. France **78** (1950), 143–151; MR **8**, 195; **12**, 479] to lattices over any finite algebraic number field and deduce applications similar to his. They also find a lower bound for the number of lattice points of their type on the boundary of a convex body, and they sketch the proof of an analogue to Minkowski's linear form theorem for forms in two variables in the quadratic field generated by $\sqrt{-5}$.

K. Mahler (Manchester)

1667:

Taniyama, Yutaka. L-functions of number fields and zeta functions of abelian varieties. J. Math. Soc. Japan **9** (1957), 330–366.

In this paper, the author (whose death, recently announced, has cut short prematurely a most promising career) not only gives a new proof, free from artificial restrictions, of his earlier theorem on the zeta-function of Abelian varieties with sufficiently many complex multiplications [cf. Y. Taniyama, Proc. internat. symposium algebraic number theory, Tokyo-Nikko (1955), pp. 31–45, Science Council of Japan, Tokyo, 1956; MR **18**, 601], but develops, in relation with that problem, a number of new ideas of far-reaching importance. Their motivation will appear most clearly if the sections of the paper are discussed in reverse order. Let k be an algebraic number-field; call $\mathfrak{g}$ the Galois group over k of its algebraic closure $\bar{k}$. Let A be an abelian variety of dimension n, defined over k; for each rational prime l, call G_l the group of the points of A whose order is a power of l, and k_l the extension of k generated by the coordinates of these points. As all the k_l are contained in $\bar{k}$, $\mathfrak{g}$ operates on G_l for every l; as G_l is the product of $2n$ groups isomorphic to the additive group of l-adic integers, this defines, for each l, a representation M_l of $\mathfrak{g}$ by l-adic matrices of order $2n$, whose kernel is the subgroup $\mathfrak{g}_l$ of $\mathfrak{g}$ attached to k_l. Now the author applies the reduction-theory (due to Shimura and himself) to this situation; this shows, first of all, that, for all prime ideals $\mathfrak{p}$ of k except for a finite number of "exceptional" ones, the reduction of A modulo $\mathfrak{p}$ determines an abelian variety of dimension n, defined over the field with $N\mathfrak{p}$ elements; it also shows that the so-called Hasse zeta-function of A can be expressed in terms of the matrices attached, in the representations M_l, to the Frobenius substitutions over k of the prime divisors in $\bar{k}$ of the non-exceptional primes of k. The author derives now from the reduction-theory some rather precise results on the latter matrices (including the fact that every non-exceptional prime $\mathfrak{p}$ is unramified in k_l for every rational prime l which is not a multiple of $\mathfrak{p}$).

The results thus obtained (§ 4, nos. 17–18) are formulated as a set of abstract conditions which the representations M_l of $\mathfrak{g}$ have to satisfy. At this point the author takes a step involving what is perhaps the most original idea of the whole paper; he considers any system (M_l) of l-adic representations of $\mathfrak{g}$, all of the same degree (l ranging over all primes) satisfying the same set of conditions. To such a system, he attaches a "zeta-function" (reducing to that of A in the case described above); and, among other results, he proves that this can be written as an infinite product of L-functions of Artin type, attached to the extensions k_l of k (abelian or not) deter-

mined by the kernels $\mathfrak{g}_l$ of the representations M_l (§ 3, no. 15, theorem 3).

Much more precise results are given for the abelian case (i.e. when all fields k_l are abelian over k); a deep analysis, based of course on class-field theory, shows that, in this case, the system (M_l) determines a "Grössencharakter of type (A_0)" [as defined by the reviewer, ibid., pp. 1–7] of the idèle-class group C_k of k; such a character has in particular the property that the Hecke L-function attached to it has algebraic coefficients; more precisely (as the author shows) the field K generated by the coefficients is either the rational field Q, or a totally imaginary quadratic extension of a totally real field. It is in fact one of the paper's main results that there is essentially a one-to-one correspondence between abelian systems of representations (M_l) of $\mathfrak{g}$ and characters of C_k of type (A_0) (§ 3, no. 11, theorem 1'); this depends upon a characterization of the latter, quite different from their original definition, by properties involving only their local behavior at $\mathfrak{p}$-adic places of k and at "sufficiently many" $\mathfrak{P}$-adic places of the field K where the coefficients of the corresponding Hecke L-function take their values (§ 1, no. 6, theorem 1). From this, it follows that the zeta-function of an abelian system of representations (M_l) of $\mathfrak{g}$ is a product of finitely many Hecke L-functions. This applies in particular to the zeta-function of an abelian variety of dimension n, defined over k, whose ring of endomorphisms over k contains an order of a number-field of degree $2n$ over Q; for it is an easy consequence of reduction-theory that the system of representations (M_l) attached to such a variety is abelian. A special case is that of any curve $ax^n+by^m=1$, since the Jacobian variety of such a curve is isogenous to a product of abelian varieties of the type described above. *A. Weil* (Princeton, N.J.)

1668:

Lehmer, Emma. Criteria for cubic and quartic residuacity. Mathematika **5** (1958), 20–29.

A prime q is kth residue modulo a prime $p=kn+1$ if and only if $q^n\not\equiv p$ and the period equation of p of degree k has a rational root mod q. From this criterion the author derives explicit and workable criteria for quartic and cubic residuacity. With the help of these criteria the author discusses special cases with ease. For instance if $p=a^2+b^2\equiv 4a^2 \pmod q$ and if $p=8m+1$, then q is always a quartic residue mod p, but if $p=8m+5$ then q is a quartic residue only if $q=12n+1$. Also if $p=L^2+27M^2$ and $L^2\equiv 5M^2 \pmod q$ then q is a cubic residue mod p. The author's criteria enable her also to derive reduction criteria which reduce the problem of testing quartic or cubic residuacity of a prime q mod p to the same problem for a smaller p_1. *H. B. Mann* (Columbus, Ohio)

1669:

Cugiani, Marco. Approssimazioni quadratiche nei domini P-adici. Ann. Mat. Pura Appl. (4) **44** (1957), 1–22.

Necessary and sufficient conditions for representation by $x^2-\alpha y^2$, where α is a given p-adic integer and x, y take p-adic integer values. Only the p-adic completions of the rationals are discussed.

J. W. S. Cassels (Cambridge, England)

1670:

Sós, Vera T. On the theory of Diophantine approximations. II. Inhomogeneous problems. Acta Math. Acad. Sci. Hungar. **9** (1958), 229–241.

The author continues the discussion of her algorithm for 1-dimensional inhomogeneous diophantine approximation [Part I, same Acta **8** (1957), 461–472; MR **20** #34], which, as she remarks, is essentially that of the reviewer [Math. Ann. **127** (1954), 288–304; MR **15**, 687]. For a pair of numbers α, β, with α irrational and β not of the shape $u+v\alpha$ (u, v integers), the algorithm gives a sequence of pairs of integers q_k, q_k' such that the lower limit of $\lambda(x)=x\|\alpha x+\beta\|$ for positive integral x is the lower limit when x is confined to be a q_k or a q_k'. (Here $\|\gamma\|$ denotes the distance of γ from the nearest integer.) [R. Descombes, Ann. Sci. Ecole Norm. Sup. (3) **75** (1956), 283–355; MR **19**, 253.] The author shows that

$$\mu_k=\min\{\lambda(q_k),\lambda(q_k')\}<\tfrac{3}{7}, \tag{1}$$

$$\min\{\mu_k,\mu_{k+1}\}<\tfrac{1}{4}, \tag{2}$$

for every k, the constants $\frac{3}{7}$ and $\frac{1}{4}$ being the best possible. She shows further, that for any $c<5^{-\frac{1}{2}}$ and any integer l there exist α, β such that

$$\mu_k>c \tag{3}$$

for l consecutive values of k. The results (1) and (2) are an analogue to classical results of Borel about homogeneous approximation; but (3) is a contrast since the lower limit of $\lambda(x)$ is at most $27/28(7)^{\frac{1}{2}}$ [Cassels, loc. cit.].

J. W. S. Cassels (Cambridge, England)

1671:

Szüsz, P. Über die metrische Theorie der Diophantischen Approximation. Acta Math. Sci. Hungar. **9** (1958), 177–193.

Let $f(x)$ be a nonnegative function of the positive integer x. The author considers the set $\mathfrak{S}_f$ of pairs (α,β) such that the inequality

$$x\|\alpha x-\beta\|\leqq f(x)$$

has infinitely many positive integer solutions x, where $\|\gamma\|$ denotes the distance of γ from the nearest integer. He proves first that if $f(x)$ is monotone decreasing, then for fixed β almost all or almost no α are such that $(\alpha,\beta)\in\mathfrak{S}_f$ according as $\sum x^{-1}f(x)$ diverges or converges. For $\beta=0$ this result is due to Khintchine (=Hinčin) [Math. Ann. **92** (1924), 115–124], and the author's proof is a development of Khintchine's. The author remarks that although Khintchine's result generalizes to n dimensions, it is not clear whether his will. Secondly, the author shows that there is no comparable result when the rôles of α and β are interchanged: more precisely, if $f(x)$ tends to 0, however slowly, there exist α such that $(\alpha,\beta)\in\mathfrak{S}_f$ for almost no β.

J. W. S. Cassels (Cambridge, England)

1672:

Schmidt, Wolfgang. Mittelwerte über Gitter. Monatsh. Math. **61** (1957), 269–276.

Let $f(X_1, X_2, \cdots, X_k)$ be a function of the k points $X_1, X_2, \cdots, X_k$ in n-dimensional Euclidean space. Let F be a fundamental domain for the space $\mathfrak{A}$ of $n\times n$ matrices A of determinant 1 relative to the integral unimodular matrices. C. L. Siegel [Ann. of Math. **46** (1945), 340–347; MR **6**, 257] introduced an invariant volume element $d\omega$ in $\mathfrak{A}$ with $\int_F d\omega=1$ and proved the case $k=1$ of the formula $(1\leqq k\leqq n-1)$

$$\int_F \sum f(Ag_1, Ag_2, \cdots, Ag_k)\,d\omega = \iint\cdots\int f(X_1, X_2, \cdots, X_k)\,dX_1dX_2\cdots dX_k,$$

the sum being taken over all sets of k linearly independent vectors $g_1, g_2, \cdots, g_k$ with integral elements. He also

stated the general result without proof. The author now gives a proof of Siegel's assertion and uses it to give a much simpler and more direct proof of an apparently more general result proved recently by the reviewer [Acta Math. **94** (1955), 249–287; MR **17**, 715].

C. A. Rogers (Birmingham)

COMMUTATIVE RINGS AND ALGEBRAS

See also 1665, 1683, 1684.

1673:

Beaumont, Ross A.; and Byrne, J. Richard. On the construction of R-modules and rings with polynomial multiplication. Pacific J. Math. **7** (1957), 1305–1317.

What ring structures does the additive group $S_1\oplus\cdots\oplus S_k$ admit if each S_i is an ideal in an integral domain R? If each component of the ring product ab is assumed to be a polynomial in the components of a and b (with coefficients in R), then these polynomials are bilinear, hence sums of monomials $ra_i^{p^r}b_j^{p^s}$ where p is the characteristic of R. Similar theorems for constructing R-module structures on this additive group; and examples.

D. Zelinsky (Evanston, Ill.)

ALGEBRAIC GEOMETRY

See also 1667.

1674:

Permutti, Rodolfo. Su certe forme a hessiana indeterminata. Ricerche Mat. **6** (1957), 3–10.

Appoggiandosi su risultati di Gordan e Noether, l'A. ritrova alcuni risultati di A. Franchetta relativi alle forme di S_4 aventi Hessiana indeterminata: tali forme possono essere rappresentate da equazioni del tipo: $f=\varphi(x_1, x_2, \theta)=0$ con $\theta=R_1x_3+R_2x_4+R_3x_5$, ove R_i denotano forme binarie dello stesso grado in x_1, x_2 e φ un polinomio nei tre argomenti x_1, x_2, θ scelto in modo che f risulti omogeneo. Tutte e sole le ipersuperficie siffatte sono coni oppure serie ∞^1 di piani tangenti ad una curva piana C^p razionale, d'ordine $p>1$, in modo che i piani della serie appartenenti ad un S_3 passante per C^p tocchino C^p nello stesso punto. Questi risultati vengono parzialmente generalizzati agli spazi superiori: in S_r con $r\geq 4$, sono ad Hessiana indeterminata tutte le forme algebriche V_{r-1} dotate di una serie ∞^1 Σ di S_{r-2} tangenti ad una varietà unirazionale V^p, d'ordine $p>1$, situata in uno spazio ad $r-2$ dimensioni, in modo che gli S_{r-2} di Σ che stiano in un iperpiano passante per V^p tocchino V^p nello stesso punto. Resta aperta la questione di vedere se anche per $r>4$ siano queste tutte le forme non coniche ad Hessiana indeterminata.

D. Gallarati (Genova)

1675:

Hutcherson, W. R.; and Childress, N. A. Surfaces obtained from involutions generated by homographies of periods three, five, and thirteen. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 41–48.

1676:

Burniat, Pol. Quelques théorèmes d'existence à propos des surfaces algébriques. Acad. Roy. Belg. Bull. Cl. Sci. (5) **44** (1958), 43–55.

In this paper the author demonstrates the existence of algebraic surfaces, for which the canonical system presents a fixed irreducible component, and gives examples of surfaces of this kind.

M. Piazzolla-Beloch (Ferrara)

1677:

Macdonald, I. G. Some enumerative formulae for algebraic curves. Proc. Cambridge Philos. Soc. **54** (1958), 399–416.

"Part I of this paper is concerned with one or more linear series on an algebraic curve; a set of points on the curve which are contained with assigned multiplicities in a set of each of the linear series is considered and, by use of Severi's equivalence relation for the united points of an algebraic correspondence with valency, formulae are derived for the number of such sets of points when the constants involved are such as to make this number finite. Part II applies these results to the problem of finding the number of multisecant and multitangent linear subspaces of the curve which in addition satisfy prescribed Schubert conditions." (From the author's summary.)

H. T. Muhly (Iowa City, Iowa)

1678:

Snapper, Ernst. Cohomology groups and genera of higher-dimensional fields. Mem. Amer. Math. Soc. no. **28** (1957), 100 pp.

Soient k un corps et $E=k(x_1, \cdots, x_n)$ un corps de fonctions algébriques sur k. Les modèles projectifs X_α de E forment, pour la relation de domination, un système projectif d'espaces topologiques, et la limite projective de ce système est la "surface de Riemann" X de E. Chaque X_α porte son faisceau L_α d'anneaux locaux, les L_α forment un système inductif de faisceaux, et leur limite inductive L s'identifie au faisceau sur X dont les fibres sont les anneaux de valuation de E.

La première partie du mémoire étudie la notion purement topologique de limite d'un système inductif de faisceaux, et les groupes de cohomologie d'une telle limite.

Dans la 2ème partie on montre que, en notant $h^q(E/k)$ la dimension sur k du q-ème groupe de cohomologie $H^q(E/k)$ de (X, L) (groupe qui est un invariant de E/k), on a $h^q(E/k)=0$ pour $q>r$ (r: degré de transcendance de E/k) et $h^0(E/k)=[k':k]$ (où k' est la fermeture algébrique de k dans E). On pose $p_g=h^r(E/k)$ ("genre géométrique"), $i=h^1(E/k)$ ("irrégularité") et, si les $h^q(E/k)$ sont tous finis, $p_a=\sum_q(-1)^q h^q(E/k)$ ("genre arithmétique").

La 3ème partie étudie le comportement des $h^q(E/k)$ par diverses transformations rationnelles (en particulier la normalisation) et montre que $h^r(E/k)$ est fini. On ignore si $h^q(E/k)$ est fini pour $q=1, \cdots, r-1$.

Enfin, étant donnée une place $\mathfrak{p}$ de E/k (triviale sur k), la réduction modulo $\mathfrak{p}$ donne un corps de fonctions algébriques G/k. Etude des relations entre les groupes $H^q(E/k)$ et $H^q(G/k)$: ils sont reliés par une suite exacte de cohomologie, où figurent aussi les groupes de cohomologie de X modulo l'ensemble A des anneaux de valuation de E/k contenus dans celui de $\mathfrak{p}$.

P. Samuel (Clermont-Ferrand)

1679:

Shimura, Gorô. Correspondances modulaires et les fonctions ζ de courbes algébriques. J. Math. Soc. Japan **10** (1958), 1–28.

Let k be a field of characteristic $\neq 2, 3$; let γ_2, γ_3 be elements of k such that $\gamma_3^3-27\gamma_3^2\neq 0$; and, in the projective plane, let E_1 be the curve (1) $X_0X_2^2=4X_1^3-\gamma_2X_0^2X_1-\gamma_3X_0^3$. Taking $(X_0, X_1, X_2)=(0, 0, 1)$ as unit element and setting in (1) $X=X_1/X_0$, $Y=X_2/X_0$ one

obtains the elliptic curve (2) $Y^2=4X^3-g_2X-g_3$. For any natural integer N, let $\mathfrak{R}_N$ be the (complete) field of elliptic modular functions of order ("Stufe") N. $\mathfrak{R}_N$ may be represented by the coordinates of points t such that $Nt=0$ on a curve (2), whose invariant is transcendental over the rationals. One defines a homomorphism λ on this elliptic curve, such that $\nu(\lambda)=p$, where p is a prime. This is called a modular correspondence of degree p. The Hecke operator T_p may be interpreted as the differential of the algebraic correspondence defined by λ. Two congruence relations are proven for modular correspondences. These are used to show that the characteristic values corresponding to the Hecke operator T_p, for cusp forms of degree 2 and for almost all primes p, do not exceed $2p^4$ in absolute value. Some of the present results generalize those of Eichler [Arch. Math. **5** (1954), 355–366; MR **16**, 116], concerning the relation between the Hecke operators and zeta-functions. *E. Grosswald* (Philadelphia, Pa.)

1680:

Burniat, Pol. Modèles de surfaces canoniques de genres $p_g=4$ et $p^{(1)}=13, 14, \cdots, 33$. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 720–730.

Dans la présente note l'A. examine une rélation due à Zappa, limitant la valeur du genre linéaire $p^{(1)}$ d'une surface algébrique en fonction de son genre arithmétique, et précisément la rélation $p^{(1)}\leq 8p_a+10$. — Pour le genre géométrique $p_g=4$, l'A. donne des modèles de surface canonique régulière de genre linéaire $p^{(1)}=13, 14, 15, \cdots, 33$, et atteint la valeur $p^{(1)}=8p_a+1=8p_g+1$. (Dans une étude précédente il avait construit, pour toute valeur de p_g, des surfaces irrégulières pour lesquelles $p^{(1)}=8p_a+9$.)
M. Piazzolla-Beloch (Ferrara)

1681:

Segre, Beniamino. Corrispondenze di Möbius e trasformazioni cremoniane intere. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **91** (1956–57), 3–19.

Eine Transformation, die durch n Funktionen in n Veränderlichen vermittelt wird, heißt eine Möbiussche Korrespondenz, wenn die Funktionaldeterminante konstant ist. Sind $n-1$ dieser Funktionen gegeben, so genügt die $n$te einer linearen partiellen Differentialgleichung. Verf. zeigt daß man das allgemeinste solche Funktionensystem durch Eliminationen und einer Quadratur, ja sogar durch Eliminationen und Differentiationen allein darstellen kann.

Die interessantesten Möbiusschen Korrespondenzen sind diejenigen, die sich in beiden Richtungen durch Polynome darstellen lassen. Verf. zeigt für $n=2$, daß sie sich als Produkt von ganzen Jonquières-Transformationen also von Transformationen des Typus $\bar{x}=x$; $\bar{y}=y+P(x)$ darstellen lassen, und zwar arbeitet er in beliebigen Körpern der Charakteristik 0. Für höhere n vermutet Verf. ein entsprechendes Ergebnis, jedoch wären bei einem Beweis ganz andersartige Methoden erforderlich. Für Körper mit einer von Nulle verschiedenen Charakteristik gilt der Satz nicht; Verf. gibt ein Gegenbeispiel. *O.-H. Keller* (Halle)

1682:

Bílek, Jan. The algebraic correspondences. Časopis Pěst. Mat. **83** (1958), 33–40. (Czech. Russian and English summaries)

The author uses the terminology of A. Weil to define an algebraic correspondence between two varieties V and W, defined over an arbitrary field, as a subvariety of $V\times W$. The principle of counting constants is established and used to show that the intersection of V^r with a primal not containing V is a bunch of varieties each with dimension $r-1$. *D. Kirby* (London)

LINEAR ALGEBRA

See also 1634, 1905, 2029, 2070, 2085.

1683:

Hodges, John H. Weighted partitions for Hermitian matrices over a finite field. Math. Nachr. **17** (1958), 93–100.

Let $q=p^n$, $p>2$ and let $\theta\in \mathrm{GF}(q^2)$, $\theta^2=v\in \mathrm{GF}(q)$, $\theta\notin \mathrm{GF}(q)$. If $\alpha=a+b\theta$, where $a, b\in \mathrm{GF}(q)$, then put $\alpha^*=a-b\theta$; if $A=(\alpha_{ij})$, put $A^*=(\alpha_{ij}^*)'$, A is Hermitian if $A^*=A$. For $a\in \mathrm{GF}(q)$, put $e(a)=e^{2\pi i t(a)/p}$, $t(a)=a+a^p+\cdots+a^{p^{n-1}}$. Now define

$$S=S(B, X, A)=\sum_{U^*AU=B} e\{\sigma(X^*U+U^*X)\},$$

where σ denotes the trace, A is a non-singular Hermitian matrix of order m, B is Hermitian of order t, X is arbitrary $m\times t$ and the summation is over all U $(m\times t)$ satisfying $U^*AU=B$. It is shown that S can be expressed in terms of generalized Kloosterman sums. A number of properties of these sums is obtained. The results are analogous to those previously obtained by the author for symmetric, skew and general square matrices [Math. Z. **66** (1956), 13–24; Arch. Math. **8** (1957), 16–22; Duke Math. J. **23** (1956), 545–552; MR **18**, 643; **19**, 6; **18**, 113].
L. Carlitz (Durham, N.C.)

1684:

Hodges, John H. Some matrix equations over a finite field. Ann. Mat. Pura Appl. (4) **44** (1957), 245–250.

Let $A, B, \cdots$ denote matrices with elements in $\mathrm{GF}(q)$; let $A(m, t)$ denote a matrix of m rows and t columns and $A(m, t; r)$ a matrix of the same dimensions and with rank r. The author considers first the matric equation

$$(*) \qquad X'A+A'X=B,$$

where $A=A(m, t; r)$, B is symmetric of order t and $X=X(m, t)$. If P, Q are non-singular matrices such that $PAQ=I(m, t; r)$ and $Q'BQ=(\beta_{ij})$, define $B_0=(\beta_{ij})$ for $r<i\leq t$, $r<j\leq t$. Then (*) is solvable if and only if $B_0=0$, in which case the number of solutions is q^e, where $e=t(m-r)+\frac{1}{2}r(r-1)$. In particular, when A is non-singular of order t, then the general solution of (*) is furnished by $X=P'(Z+\frac{1}{2}Q'BQ)Q^{-1}$, where $PAQ=I$ and Z is an arbitrary skew matrix of order t.

Results similar to the above are also obtained for the equations $X'A-A'X=B$, where B is skew, and $U^*A+A^*U=B$, where B is Hermitian (relative to $\mathrm{GF}(q^2)$). Finally, the equation $UA_1+A_2V=B$ is discussed.
L. Carlitz (Durham, N.C.)

1685:

Labutin, D. N. On the rank of a matrix. Kabardin. Gos. Ped. Inst. Uč. Zap. **8** (1955), 29–32. (Russian)

1686:

Mirsky, L. Proofs of two theorems on doubly-stochastic matrices. Proc. Amer. Math. Soc. **9** (1958), 371–374.

Let y and x be real n-vectors. Let $y<x$ mean $\sum_{i=1}^k \bar{y}_i\leq \sum_{i=1}^k \bar{x}_i$, $1\leq k\leq n-1$, $\sum_{i=1}^n \bar{y}_i=\sum_{i=1}^n \bar{x}_i$; here $\bar{y}_1\geq\bar{y}_2\geq\cdots\geq \bar{y}_n$, $\{\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n\}=\{y_1, y_2, \cdots, y_n\}$, and similarly for the x's. A real matrix D is called doubly stochastic (d.s.) if the entries are non-negative with column and row sums

being equal to one. Simplified proofs are given for the following two well-known theorems. Th. I: $y<x$ is a necessary and sufficient condition for the existence of a d.s. matrix D such that $y=Dx$. Th. II: The set of d.s. matrices is identical with the convex hull of the permutation matrices. *S. Sherman* (Philadelphia, Pa.)

1687:

Schreiber, Shmuel. On a result of S. Sherman concerning doubly stochastic matrices. Proc. Amer. Math. Soc. **9** (1958), 350–353.

The d.s. matrices [defined as in #1686 above] may be ordered by defining $P_1<P_3$ to mean there exists a d.s. P_2 such that $P_1=P_2P_3$. Real n-vectors may be ordered by letting $a<b$ mean there exists a d.s. P such that $a=Pb$. The reviewer incorrectly attempted to prove that $P_1<P_3$ if and only if $P_1a<P_3a$ for each real vector a [same Proc. **3** (1952), 511–513; **5** (1954), 998–999; MR **14**, 346; **16**, 326]. The equivalence is justified when P_3 has an inverse. The ordering is given a geometric interpretation. *S. Sherman* (Philadelphia, Pa.)

1688:

Lehrer, Yehiel. On functions of matrices. Rend. Circ. Mat. Palermo (2) **6** (1957), 103–108.

If A is a matrix whose minimal equation has simple characteristic roots $\lambda_1, \cdots, \lambda_m$ and $f(A)$ is defined formally by a power series or by a Fourier series, then

$$f(A)=\sum_{j=0}^{m-1}\frac{c_j}{v}A^j,$$

where v is the Vandermonde determinant

$$v=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \cdot & \cdot & \cdots & \cdot \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_m^{m-1} \end{vmatrix}$$

and c_j is obtained from v by substituting the row $f(\lambda_1), \cdots, f(\lambda_m)$ for its $(j+1)$th row. The author also indicates the form of the result (with confluent Vandermonde determinant, etc.) when there are multiple characteristic roots of the minimal equation. *H. S. A. Potter* (Aberdeen)

1689:

Stojaković, Mirko. Quelques remarques sur les hypermatrices. Acad. Serbe Sci. Publ. Inst. Math. **11** (1957), 33–42.

Les hypermatrices sont des matrices sur un anneau commutatif R de matrices carrées sur un corps commutatif K. Elles peuvent donc être regardées comme matrices sur R et comme matrices sur K. L'Auteur établit, de manière purement algébrique, que pour toute hypermatrice carrée, on a

$$\det_K H=\det_K \det_R H.$$

L'Auteur utilise systématiquement une définition axiomatique des déterminants qu'il a donnée antérieurement [Bull. Soc. Math. Phys. Serbie **6** (1954), 40–55; MR **17**, 6]. Il retrouve un résultat d'Egervary sur l'adjointe des hypermatrices [Acta Sci. Math. Szeged **15** (1954), 211–222; MR **16**, 327]. Il termine en généralisant la notion de composabilité des matrices rectangulaires. *G. Papy* (Bruxelles)

1690:

Börsch-Supan, Wolfgang. Obere Schranken für den grössten Eigenwert eines vollstetigen selbstadjungierten Operators. Math. Ann. **134** (1958), 453–457.

The largest eigenvalue $\lambda_{\max}$ of a completely continuous self-adjoint operator A in a Hilbert space H is bounded above by the following means.

The Rayleigh-Ritz lower bound λ_1 for $\lambda_{\max}$ and the corresponding normalized eigenvector e_1 in an n-dimensional subspace R_{12} of H are computed exactly. Let c_{22} denote an upper bound for the second Rayleigh-Ritz eigenvalue. Let c_{13}, c_{23}, and c_{33} denote bounds for $|(Ae_1, e_3)|$, $|(Ae_2, e_3)|$, and $|(Ae_3, e_3)|$, respectively, where e_1 is defined above, while e_2 and e_3 are arbitrary unit vectors in $R_{12}\ominus e_1$ and $H\ominus R_{12}$, respectively.

Then the largest root μ of the cubic equation

$$(\lambda_1-\mu)(c_{22}-\mu)(c_{33}-\mu)-(\lambda_1-\mu)c_{23}^2-(c_{22}-\mu)c_{13}^2=0$$

is an upper bound for $\lambda_{\max}$.

It is shown that for $c_{22}<\lambda_1$ this bound is better than that of the reviewer [Inst. for Fluid Dynamics and Applied Math., Univ. of Md. Tech. Note **BN-41**, 1954] which, however, also gives bounds for the lower eigenvalues. (The author's result was found independently of that of the reviewer and communicated to the latter in 1955.)

The bound given here reduces to that of F. Koehler [Pacific J. Math. **7** (1957), 1391–1404; MR **19**, 1198] in the case $n=2$, but is not as good for larger n. (Koehler also gives bounds for lower eigenvalues.) *H. F. Weinberger* (College Park, Md.)

1691:

Börsch-Supan, W.; und Bottenbruch, H. Eine Methode zur Eingrenzung sämtlicher Eigenwerte einer hermiteschen Matrix mit überwiegender Hauptdiagonale. Z. Angew. Math. Mech. **38** (1958), 169–171.

A method for bounding the eigenvalues $\lambda_1\geqq\lambda_2\geqq\cdots$ of an hermitian $n\times n$ matrix a_{ij} is presented. It is an immediate consequence of the min max definition of eigenvalues that if μ_k is the largest eigenvalue of the principal minor a_{ij} with $i, j=k, k+1, \cdots, n$, then $\lambda_k\leqq\mu_k$. An application of the maximum characterization of μ_k together with Schwarz's inequality leads to the recursive inequality

$$\mu_k\leqq\tfrac{1}{2}(a_{kk}+\mu_{k+1}+\{(a_{kk}-\mu_{k+1})^2+4Q_k^2\}^{\frac{1}{2}}),$$

where $Q_k^2=\sum_{j=k+1}^{n}|a_{kj}|^2$. Consequently, if the numbers D_k are computed from the recursion

$$D_k=\tfrac{1}{2}(a_{kk}+D_{k+1}+\{(a_{kk}-D_{k+1})^2+4Q_k^2\}^{\frac{1}{2}}),$$
$$D_n=a_{nn},$$

D_k is an upper bound for λ_k. In a similar manner, lower bounds d_k for the λ_k are given by the recursion

$$d_k=\tfrac{1}{2}(a_{kk}+d_{k-1}+\{(a_{kk}-d_{k-1})^2+4q_k^2\}^{\frac{1}{2}}),$$
$$d_1=a_{11},\quad q_k^2=\sum_{j=1}^{k-1}|a_{kj}|^2.$$

These upper and lower bounds are accurate provided that (1) $a_{ii}^2>>\sum_{j\neq i}|a_{ij}|^2$, (2) $a_{11}\geqq a_{22}\geqq\cdots\geqq a_{nn}$, and (3) $|a_{k,k+1}|^2$ is not too small relative to Q_k^2 and q_k^2. The second condition can be brought about by a rearrangement of the matrix. A method based on another paper of Börsch-Supan [#1690 above] is indicated for improving the results when the third condition is violated. *H. F. Weinberger* (College Park, Md.)

1692:

Vidav, Ivan. Construction de quelques formes linéaires positives. Acad. Serbe Sci. Publ. Inst. Math. **11** (1957), 67–72.

Let A be a *-algebra with unity 1 over the complex field generated by three self-adjoint elements a, b, c for which the relations $bc-cb=ia$, $ca-ac=ib$, $ab-ba=ic$,

$i=\sqrt{-1}$ hold. A real number λ is said to be an eigenvalue of the self-adjoint element $x(\in A)$, if there exists a linear form $f(z)$ satisfying $f(zz^*)\geq 0$ for every z $(\in A)$ and $f[(x-\lambda\cdot 1)^2]=0$. The author determines all eigenvalues of the self-adjoint element $a^2+b^2+c^2$, and he constructs the linear forms $f(z)$ corresponding to the eigenvalues. This investigation is of some interest also from the point of view of quantum mechanics. *A. Kertész* (Debrecen)

1693:

Hämisch, Werner. Tensorielle Produkte regulärer Linearformenmoduln. Math. Ann. **134** (1957), 101–113.

Il est classique de définir l'algèbre tensorielle d'un espace vectoriel finidimensionnel en utilisant le produit ordinaire de formes linéaires et multilinéaires sur le dual, ce qui est licite grace à la réflexivité. L'Auteur étudie sous quelles conditions la méthode reste applicables pour des anneaux-modules unitaux. Il montre qu'il en est ainsi pour les modules réguliers, c'est-à-dire les sous-duaux possédant une base en dualité avec une famille libre du module original. Un module sera dit totalement régulier s'il est un sous-dual dont toutes les bases jouissent de la propriété précédente. La portée de la théorie provient du fait que tout module libre est isomorphe à un module totalement régulier, et que tout sous-dual régulier possédant une base finie est totalement régulier.

L'auteur développe ensuite de manière fort élégante une théorie du calcul tensoriel suivant ces idées.

G. Papy (Bruxelles)

1694:

Climescu, Al. Une définition axiomatique des déterminants. Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), no. 3–4, 1–7. (Romanian. Russian and French summaries)

Let $A=[a_{ij}]$ be a matrix of order n with elements in a (commutative) field. The theory of determinants of such matrices is developed from the axioms: I. If $n=1$, then $|A|=a_{11}$. II. $|AB|=|A||B|$. III. If $A=\begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$ or $A=\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, where B and D are square, then $|A|=|B||D|$. Assuming existence, the usual elementary properties, and uniqueness, follow without difficulty. The existence of determinants of order n is established by showing that the determinant — if it exists — is given by the usual $n!$-term formula. *J. H. Williamson* (Cambridge, England)

1695:

Gáspár, Gyula. A new proof of the Laplace expansion formula. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 491–495. (Hungarian)

The author sketches a proof of the formula mentioned in the title. The proof makes use only of those properties which he used to characterize determinants [Publ. Math. Debrecen **3** (1954), 257–260, MR **17**, 338]. The idea of the proof is shown on a special case. *J. Aczél* (Debrecen)

1696:

***Cohn, P. M. Linear equations.** Library of Mathematics. The Free Press, Glencoe, Ill.; Routledge and Kegan Paul Ltd., London; 1958. viii+74 pp. $1.25; 5s.

The exposition has been limited to basic notions, explained as fully as possible, with detailed proofs.

From the preface

ASSOCIATIVE RINGS AND ALGEBRAS

See also 1808.

1697:

Nobusawa, Nobuo. A note on Galois extensions of division rings. Math. J. Okayama Univ. **7** (1957), 179–183.

Let A be a division ring and B a division subring. A is said to be locally finite over B if $[B(\Delta):B]_l$ is finite for any finite subset Δ of A. $V(C)$ denotes the centralizer in A of any subset C of A. Continuing his study of the Galois theory of A/B [Osaka Math. J. **7** (1955), 1–6; **8** (1956) 43–50; MR **16**, 1084; **18**, 7] the author shows the following. (1) If A is locally finite over B, then A is also locally finite over $V(V(B))$. (2) Let A/B be locally finite and Galois (i.e. B is the fixed ring of some group of automorphisms of A). Then for any division subring B' containing B and finite on the left over B, the extension A/B' is also Galois. (3) Let A/B be locally finite and Galois and write B^0 for $V(V(B))$. Then for any division subring C containing B we have B^0C/C is outer Galois and locally finite with Galois group isomorphic to that of $B^0/B^0\cap C$.

A. Rosenberg (Evanston, Ill.)

1698:

Širšov, A. I. On rings with identity relations. Mat. Sb. N.S. **43(85)** (1957), 277–283. (Russian)

The heights of every associative algebra with identity relations of degree n are locally bounded (relative to a set of words of degree less than n on an arbitrary set of generators). In an associative ring with finitely many generators and admissible identity relations of degree n, if every product of less than n generators is algebraic over a subring Z_1 of the center of the ring, then the ring is finite over Z_1. An alternative ring with finitely many generators and essential identity relations is locally finite over an arbitrary subring of its center, over which J-monomials of r_2-words on its generators are algebraic. See the result of Kaplansky on local finiteness of certain algebras [Trans. Amer. Math. Soc. **68** (1950), 62–75; MR **11**, 317].

R. A. Good (College Park, Md.)

1699:

Patterson, E. M. On right-multiplication algebras. Quart. J. Math. Oxford Ser. (2) **8** (1957), 260–271.

Let V_n be an n-dimensional vector space over a field F, let A be a linear algebra over V_n and let $R(A)$ be the right multiplication algebra of A. The genus of A is the difference between its dimension and the least number of linearly independent elements whose products generate A. [See M. S. Knebelman, Ann. of Math. (2) **36** (1935), 46–56.] The author has previously studied algebras of genus zero (A) [J. London Math. Soc. **31** (1956), 326–331; MR **18**, 109] and algebras of genus one (B) [ibid. **32** (1957), 88–94; MR **18**, 638]. In the present paper he assumes that the integer n is never less than two and that the underlying field F is neither GF(2) nor GF(3). Principal results follow. (I) Let C be an n-dimensional associative algebra of genus zero, consisting of transformations of V_n into itself, with multiplication being defined in the usual way. Then, if A and A' are two algebras over V_n such that $R(A)=R(A')=C$, then there exists an automorphism β of V_n such that, for all x, y in V_n, $x\circ y=x\beta(y)$, where $x\circ y$ denotes multiplication in A' and xy denotes multiplication in A. (II) Let C be an $n+1$ dimensional associative algebra of genus one, with unit element, consisting of transformations of V_n into itself. Then, if A and A' are as in (I), there exist an automorphism β of V_n onto itself and a linear function h (V_n into F) such that $x\circ y=xh(y)+x\beta(y)$,

with multiplication as in (I). The author remarks that the converse of (II) is false and devotes the remainder of the paper to more technical theorems of which one example will be given. If $R(A)$ is of genus one and has a unit element, the author proved in reference (B) that there exist linear functions p and q such that multiplication in $R(A)$ is given by $UV=Up(V)+Vq(U)-p(V)q(U)I$, where I is the identity. Theorem: Let A be an algebra over V_n such that $R(A)$ is of dimension $n+1$, of genus one, and contains the identity. If A' is an algebra over V_n with multiplication given by $x\circ y=xh(y)+x\beta(y)$, where β is an automorphism and h is a linear function, then $R(A')=R(A)$ unless $h(y)=-p(Y'')$ or $h(y)=-q(Y'')$, Y'' being the transformation sending x into $x\beta(y)$. Moreover, if $p=q$, so that $R(A)$ is commutative, and if $h(y)=-p(Y'')$, then $R(A')$ is the zero algebra of dimension n. If $p\neq q$ and $h(y)=-p(Y'')$ or $-q(Y'')$, then $R(A')$ is an algebra of dimension n and genus zero.

R. L. San Soucie (Buffalo, N.Y.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See also 1698.

1700:

Patterson, E. M. On certain classes of linear algebras of genus one. Proc. Roy. Soc. Edinburgh. Sect. A. **65** (1958), 63–71.

For terminology, see the review #1699 above. In the present paper the author uses reference (B) of that review and a more recent paper [Proc. London Math. Soc. (3) **7** (1957), 467–480; MR **19**, 836] to prove the following: If A is a Jordan algebra of dimension $n\geq 4$ and genus one, then multiplication in A is of one of the following types: (i) $xy=-\theta(y,c)x-\theta(x,c)y+\theta(x,y)c$, where θ is a symmetric bilinear function (s.b.f.); (ii) $xy=\theta(y,c)x+\theta(x,c)y+\theta(x,y)c$, where θ is an s.b.f. such that $\theta(c,c)=0$; (iii) $xy=xg(y)+yg(x)+c[h(x)h(y)-g(x)g(y)]$, where g and h are linear functions such that $g(c)=0$ and $h(c)=1$. In each case, c is a fixed element of the algebra. Conversely any algebra of one of these types is a Jordan algebra of genus one. If, in addition to the listed hypotheses, A is simple, then case (i) holds, with θ a non-singular s.b.f. such that $\theta(c,c)\neq 0$. The author notes that a comparison with A. A. Albert's classification of simple Jordan algebras [Ann. of Math. (2) **48** (1947), 546–567; MR **9**, 77] shows that, for algebraically closed fields, the class of simple Jordan algebras of dimension $n\geq 4$ and genus one coincides with the class of simple Jordan algebras of degree two.

R. L. San Soucie (Buffalo, N.Y.)

1701:

Kostrikin, A. I. Lie rings satisfying the Engel condition. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 515–540. (Russian)

Details are presented for results announced earlier [Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 580–582; MR **18**, 188]. For a Lie ring with the Engel condition $E_{n,p}$, its radical is a locally nilpotent ideal and the quotient-ring modulo its radical has a radical that is zero. For $n=4, 5, 6$, a Lie ring satisfying $E_{n,p}$ is shown to be locally nilpotent. {The aforementioned review, by mistake, listed 3 as one of the values of n treated by the author.} Application to the restricted Burnside problem with exponent 5 or 7 is made; thus [see P. Hall and G. Higman, Proc. London Math. Soc. (3) **6** (1956), 1–42; MR **17**, 344] the restricted Burnside conjecture for q generators and exponent m is substantiated for arbitrary q if m is any one of the numbers 2, 3, 4, 5, 6, 7, 10, 12, 14, 15, 20, 21, 28, 35. Compare the contemporary work of G. Higman [Proc. Cambridge Philos. Soc. **52** (1956), 381–390; MR **18**, 377].

R. A. Good (College Park, Md.)

1702:

Simoniya, V. T. Representations of semi-simple Lie algebras of rank 2. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 223–407. (Russian)

As is well known, an irreducible representation of a semi-simple Lie algebra is determined by its highest weight. In this paper explicit forms of irreducible representations of the algebras A_2, $B_2(=C_2)$, D_2, and G_2 are given in terms of their highest weights. For the algebras A_n, B_n, D_n this has already been done by Gelfand and Cetlin [Dokl. Akad. Nauk SSSR (N.S.) **71** (1950), 825–828, 1017–1020; MR **12**, 9; **11**, 639], but the expressions given in the present paper are all rational. The major part of the paper is devoted to the algebra G_2.

R. Ree (Vancouver, B.C.)

1703:

Climescu, Al. La représentation par des matrices groupales du groupoïde multiplicatif d'une algèbre non-associative. Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), no. 3–4, 9–18. (Romanian. Russian and French summaries)

A denotes a linear algebra of finite order over the real field F with the following properties. There is a fixed non-zero element e of A and a mapping $x\to\lambda(x)$ of A on F such that for all elements x, y, z of A, $e(ex)=x$, $e(xy)=(ex)(ey)$, $x(yz)=e[(xy)z]$, $xe=\lambda(x)e$. Among results proved are the following: $e^2=e$; $xy=\lambda(x)ey$; $x(yz)=\lambda(x)\lambda(y)z$; $(xy)z=\lambda(x)\lambda(y)ez$; $x\to\lambda(x)$ is a homomorphism of A {hence in the reviewer's terminology [Proc. London Math. Soc. (2) **52** (1951), 241–252; MR **12**, 798] A is a non commutative train algebra with weight function $\lambda(x)$ having left and right ranks at most 3}; the isotope of A obtained by replacing multiplication by $a\cdot b=(ea)(eb)$ is associative; the basis can be chosen to consist of a set of elements e_i for which $ee_i=e_i$ and a set (possibly null) of elements f_j for which $ef_j=-f_j$.

Examining the algebras A of order 2 over F, the author finds two which are associative {actually these are isomorphic but correspond to different mappings $\lambda(x)$}; and a third which is nonassociative consisting of elements $\alpha e+\beta f$ $(\alpha, \beta\in F)$ where $e^2=e$, $ef=-f$, $fe=f^2=0$. For the latter, he determines all groupal matrices $M(\alpha,\beta)$, that is to say matrices which represent the algebra as regards multiplication only.

I. M. H. Etherington (Edinburgh)

HOMOLOGICAL ALGEBRA

See also 1661, 1679, 1712.

1704:

Morita, Kiiti; and Tachikawa, Hiroyuki. Character modules, submodules of a free module, and quasi-Frobenius rings. Math. Z. **65** (1956), 414–428.

Let U be a double module over a ring A. For a left A-module L, the right A-module $\mathrm{Hom}_A(L, U)$ is called the U-character module of L; similarly for a right A-module. Let a left A-module L and a right A-module R form an

orthogonal pair with respect to an inner product into U. Under the assumption that the U-character module of a simple left (right) A-module is simple and that one of L, R has a composition series, the paper shows that each of L, R is the U-character module of the other and the submodules of L, R are in 1-1 duality. The proof is a generalization of an argument in Nakayama, Ann. of Math. (2) **42** (1941), 1–21 [MR **2**, 344] and provides a simultaneous foundation for the duality in the last paper and Hall's [ibid. **40** (1939), 360–369] theorem. [For a somewhat similar, though different, foundation of the theory of quasi-Frobenius rings, cf. Kasch, Math. Ann. **127** (1954), 453–474; MR **16**, 7.] Next, let A satisfy the minimum condition and let $A^{(n)}$ be the free left A-module with n generators. It is shown that an isomorphism of two residue-modules $A^{(n)}/L$, $A^{(n)}/L'$ can always be induced by an automorphism of A, and implies in particular $L \cong L'$; this generalizes special cases in Gaschütz, Math. Z. **60** (1954), 274–286 [MR **16**, 671], and Kawada, J. Math. Soc. Japan **9** (1957), 374–380 [MR 20 #3191]. The paper derives further a theorem which states essentially that A is left A-injective if and only if it is a quasi-Frobenius ring [Ikeda, Osaka Math. J. **4** (1952), 203–209; MR **14**, 719], and discusses some of its consequences.

T. Nakayama (Nagoya)

1705:

Hattori, Akira. On Λ-injectivity (problem 6.3.19). Sûgaku **8** (1956/57), 208–209. (Japanese)

Let Λ be a (non-commutative) ring ($\ni 1$), and A, B be left Λ-modules. Let the "tensor product" $A \otimes B$ of A, B over Λ be defined by the bilinearity and $\lambda a \otimes b = a \otimes \lambda b$ ($\lambda \in \Lambda$, $a \in A$, $b \in B$); it is a left Λ-module. The note shows by a counter example the incorrectness of the following proposition: (I) If A, B are Λ-injective, $A \otimes B$ is Λ-injective. Similarly $A \otimes B$ with Λ-projective A, B need not be Λ-projective. The note remarks also that (I) holds good in case Λ is a (commutative) integrity domain.

T. Nakayama (Nagoya)

GROUPS AND GENERALIZATIONS

See also 1642, 1648, 1701, 2120.

1706:

Balcerzyk, S.; and Mycielski, Jan. Some theorems on the representations of free products. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1029–1030, LXXXVI. (Russian summary)

Announcement that (1) if G_ξ ($\xi <$ ordinal number α) is a sequence of groups, each isomorphic to a subgroup of the rotation group R of E^3, and $\sum_{\eta \le \xi} \operatorname{card} G_\eta < 2^{\aleph_0}$ any $\xi < \alpha$, then the free product $\prod^* G_\xi$ is isomorphic to a subgroup of R; (2) two elements of R with different axes, but common rotation angle φ, freely generate a free subgroup except for denumerably many φ depending on the axes; (3) if the index set T and groups G_t, $t \in T$, have powers $\le \aleph_{\alpha+1}$, $\aleph_\alpha$ respectively, then $\prod^* G_t$ is isomorphic to a subgroup of the permutation group of a set of power $\aleph_\alpha$.

1707:

Dlab, Vlastimil. A note on the theory of divisible abelian groups. Czechoslovak Math. J. **8(83)** (1958), 54–61. (Russian. English summary)

In this article, systems of generators S of a group G are classified as follows: System S is reducible if $S \setminus g$ generates G for some $g \in S$; s-reducible if $S \setminus g$ generates G for every $g \in S$; h-reducible if every subset $S' \subset S$ such that $\{S'\} = G$ is reducible; s-h-reducible if $S' \subset S$, $\{S'\} = G \to S'$ is s-reducible. The chief contribution of the paper (in spite of its title) is the presentation of an example showing that these properties of a generating system in no way characterize an abelian group. A particular group is exhibited having systems of generators of all logically possible types: irreducible; reducible but not s- or h-reducible; s- but not h-reducible; h- but not s-reducible; s- and h- but not s-h-reducible; s-h-reducible.

J. L. Brenner (Palo Alto, Calif.)

1708:

Čunihin, S. A. Π-factorization of finite groups. Mat. Sb. N.S. **43(85)** (1957), 49–66. (Russian)

Details are given for results announced earlier [Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 397–399; MR **18**, 110].

R. A. Good (College Park, Md.)

1709:

***Brauer, Richard. On the structure of groups of finite order.** Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 209–217. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

In recent years the author and others have applied character theory to prove important results on the structure of finite groups. The present address describes some of this work and poses further problems.

The author first considers the existence of relatively large subgroups of a group. Theorem 1: A group G of even order $g > 2$ contains a proper subgroup H of order $h > \sqrt[3]{g}$. (It is easy to prove that a soluble group G of composite order g contains a proper subgroup of order $h \ge \sqrt{g}$.) Theorem 2: Let m denote the number of involutions (= elements of order ≤ 2) in a group G of even order g and set $n = g/m$. There exists a subgroup $H \ne G$ of G such that the index $t = (G:H)$ either is 2 or satisfies $t \le n(n+2)/2$. The proofs are elementary and depend on estimates for the number of pairs of involutions in G.

It follows easily from theorem 2 that there exist only a finite number of simple groups G which contain an involution J whose normalizer is isomorphic to a given group N. The next problem posed is to find the possible G for given N. Several methods are proposed, the most important being to apply the theory of p-blocks. (In this connection it is conjectured that the number, s, of characters in a p-block of defect d satisfies $s \le p^d$; this has been proved for $d \le 2$, but the best known result for larger d is $s < \frac{1}{4} p^{2d}$.) The simple groups of order $g = 4g'$ (g' odd) are instanced; by considering 2-blocks it has been proved that $g = 32 v^3 f_1 f_2 f_3 / (f_1 + 1)(f_2 \pm 1)(f_3 - 1)$, where $4v$ is the order of the normalizer of an involution in G and the f_i are degrees of certain irreducible characters of G. There are infinitely many such simple groups, viz., groups $\mathrm{PSL}(2, q)$ for suitable q.

The author next gives group-theoretical characterizations of the simple groups $\mathrm{PSL}(2, q)$ and $\mathrm{PSL}(3, q)$. Theorem 3: Suppose that G is a group of even order g such that if two cyclic subgroups A, B of even order have an intersection different from $\{1\}$, then A, B are both subgroups of a cyclic subgroup of G. If G does not have a normal subgroup of index 2, then G is a group $\mathrm{PSL}(2, q)$ (with $q > 2$). Theorem 4: Suppose that the group G of even order satisfies the conditions: (I) G contains an involution J whose normalizer $N(J)$ is isomorphic to $\mathrm{GL}(2, q)$; (II) if C is an element $\ne 1$ of the centre of $N(J)$,

the normalizer of C in G is $N(J)$; (III) G is its own commutator group. If $q=3$ or 11 (mod 12) and $q\neq 3$, then G is isomorphic to PSL(3, q). If $q=3$, there is the additional case that G can be the simple Mathieu group of order 7912.

In the proof of theorem 4, a characterization of PSL(3,q) by axioms on its involutions is used; these axioms come from the rôle of PSL(3, q) as group of collineations of a projective plane. This leads the author to consider a kind of 'geometry' in groups G. He defines the distance $d(x, y)$ between two elements x, y of $G_1=G-\{1\}$ as follows. If there exists a chain $x_0=x, x_1, \cdots, x_r=y$ of elements x_i of G such that any consecutive elements x_{i-1}, x_i commute, then $d(x, y)$ is the length, r, of the shortest such chain; if no such chain exists, then $d(x, y)=\infty$. G is then a metric space in which infinite distances are admitted. Finally, two fairly easily proved results in this geometry are cited. Theorem 5: Let G be a group of even order and $x(\neq 1)$ a real element of G (a real element is one conjugate to its inverse). If the distance $d(x, M)$ of x from the set, M, of involutions is larger than 3 then $d(x, M)$ is infinite. In this case, the normalizer $N(x)$ is an abelian group H whose order is relatively prime to its index $(G:H)$. The group H consists only of real elements and is the normalizer of each of its elements different from 1. Theorem 6: If the group G of even order contains more than one conjugacy class of involutions, then any two involutions of G have distance at most 3.

G. E. Wall (Sydney)

1710:

Tyškevič, R. I. On nilpotent linear groups. Mat. Sb. N.S. **42(84)** (1957), 441–444. (Russian)

The note extends results obtained by D. A. Suprunenko [Mat. Sb. N.S. **31(73)** (1952), 353–358; **35(77)** (1954), 501–512; Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **1953**, no. 15, 3–6; MR **14**, 447; **16**, 793; **17**, 456]. Let P be an algebraically closed field, M its multiplicative group, GL(n, P) the general linear group of degree n over P. Let G be an irreducible nilpotent subgroup of GL(n, P) of class $k>1$ and maximal among the subgroups of that class; and $Z_0=1<Z_1=M<Z_2<\cdots<Z_{k-1}<Z_k=G$ the upper central series of G. (i) The only irreducible group in this series is G; (ii) if $k>2$, then Z_2 is abelian; and (iii) if the index of Z_1 in Z_2 is n, then Z_{k-1} is abelian, so that G is metabelian.

K. A. Hirsch (London)

1711:

Obata, Morio. On subgroups of the orthogonal group. Trans. Amer. Math. Soc. **87** (1958), 347–358.

D. Montgomery and H. Samelson [Ann. of Math. (2) **44** (1943), 454–470; MR **5**, 60] have proved that the connected Lie subgroups G of the orthogonal group O(n) satisfying $\dim O(n)>\dim G\geq\dim O(n-1)$ are all conjugate to the standard subgroup SO($n-1$), except for $n=4,8$. Furthermore, they proved there are no subgroups G such that $\dim O(n-1)>\dim G>\dim O(n-2)+\dim O(2)$ with a finite (but undetermined) number of exceptional values for n. The result of the present paper supplements these data: if $n\geq 14$, and $\dim O(n-1)>\dim G>\dim O(n-3)+\dim O(3)$, then G is conjugate to either of the standard subgroups SO($n-2$) or SO($n-2$)$\times$SO(2). The method used is largely due to H. C. Wang and K. Yano [Trans. Amer. Math. Soc. **80** (1955), 72–92; MR **17**, 407], and is based on a detailed study of irreducibility, absolute irreducibility and unitary symplectic representations.

A. Nijenhuis (Seattle, Wash.)

1712:

Griffiths, H. B. On limits of systems of groups. Proc. Amer. Math. Soc. **9** (1958), 118–129.

A unified definition (and generalization) of direct and inverse limits is given and applied to give a unified proof of the following theorem on direct limits and its dual for inverse limits: Let f be a homomorphism of a directed system of groups and homomorphisms $\{A^m, a_m{}^n\}$ to a similar directed system $\{B^m, b_m{}^n\}$, and suppose that, for each $m<n$, there is a homomorphism $\varphi_{mn}:B^m\to A^n$ satisfying $f_m\varphi_{mn}=b_m{}^n$ and $\varphi_{mn}f_n=a_m{}^n$. Then f induces an isomorphism of the direct limits of A and B.

D. Zelinsky (Evanston, Ill.)

1713:

Livingstone, D. Proof of a theorem discovered by Murnaghan. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 618–619.

Murnaghan [same Proc. **41** (1955), 514–515; MR **17**, 12] stated without proof that the rth compound matrix of the matrix representation corresponding to the partition $[n-1, 1]$ of the symmetric group of order $n!$, yielded exactly the representation corresponding to the partition $[n-r, 1^r]$. The author gives a proof. It may be remarked that the reviewer has proved the same theorem incidentally and independently [$\neq$1714 below, pp. 6 and 15].

D. E. Littlewood (Bangor)

1714:

Littlewood, D. E. The inner plethysm of S-functions. Canad. J. Math. **10** (1958), 1–16.

If $[\mu]$ and $[\nu]$ are two irreducible representations of S_m and S_n, then the outer product $[\mu]\cdot[\nu]$ is a representation of S_{m+n} with character $((m+n)!/m!n!)\chi^\mu\chi^\nu$. The corresponding tensor representation of GL(d) of rank $m+n$ has character the product of Schur functions $\{\mu\}\{\nu\}$. The product of n equal factors $\{\mu\}$ splits into a sum of 'outer plethysms':

$$\{\mu\}\{\mu\}\cdots\{\mu\}=\sum_\nu f^\nu\{\mu\}\otimes\{\nu\}.$$

If instead of outer products we consider the 'inner' or Kronecker product $[\mu]\times[\nu]$ of two representations $[\mu]$, $[\nu]$ of S_m with character $\chi^\mu\chi^\nu$, we can again reduce the corresponding product of n equal Schur functions

$$\{\mu\}\cdot\{\mu\}\cdot\cdots\cdot\{\mu\}=\sum_\nu f^\nu\{\mu\}\odot\{\nu\}$$

and call $\{\mu\}\odot\{\nu\}$ an 'inner plethysm'.

The evaluation of $\{\mu\}\otimes\{\nu\}$ and $\{\mu\}\odot\{\nu\}$ has been the subject of much effort of recent years and it is useful to have relations such as

$$(A\otimes B)\otimes C=A\otimes(B\otimes C),$$
$$(A\odot B)\odot C=A\odot(B\odot C),$$

set out in § 1 of the present paper.

In § 2 the author proves the following four theorems:

$$(\{n-1\}\{1\})\odot\{r\}=\sum\{n-\mu_1\}\{\mu_1-\mu_2\}\cdots\{\mu_j\},\ \sum\mu_i=r,$$
$$(\{n-1\}\{1\})\odot\{1^r\}=\sum\{n-r, 1^r\},$$
$$(\{n-r\}\{r\})\odot\{2\}=\sum(\{i\}\otimes\{2\})\{r-i\}\{n-r-i\},$$
$$(\{n-r\}\{r\})\odot\{1^2\}=\sum(\{i\}\otimes\{1^2\})\{r-i\}\{n-r-i\},$$

which are helpful in the evaluation of $\{\lambda\}\odot\{\mu\}$, but the results are very complicated when stated with any degree of generality (§§ 4, 5).

G. de B. Robinson (Toronto, Ont.)

1715:

Littlewood, D. E. Products and plethysms of characters with orthogonal, symplectic and symmetric groups. Canad. J. Math. **10** (1958), 17–32.

This second paper provides proofs of results due to

Murnaghan [Proc. Nat. Acad. Sc. **41** (1955), 515–518; MR **17**, 12] in the reduction of the Kronecker product $[\mu]\times[\nu]$ of two representations $[\mu]$, $[\nu]$ of S_m [see #1714 above], and so provides an alternative method for obtaining the reduction of the inner plethysm $\{\mu\}\odot\{\nu\}$. Though fundamentally involved, a result such as

$$\{5, 1\}\odot\{2, 1\}=\{3, 2, 1\}+\{4, 2\}+\{4, 1^2\}+\{5, 1\}$$

can be derived systematically without undue labour.

G. de B. Robinson (Toronto, Ont.)

1716:

Adyan, S. I. The role of the cancellation law in presenting cancellation semi-groups by means of defining relations. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 1191–1194. (Russian)

Let Π, Π_L, Π_R, Π_2 denote finitely presented semigroups, left-, right-, and two-sided cancellation semigroups, respectively. Are the cancellation laws superfluous? Every Π_L, Π_R, Π_2 can also be given as an ordinary semigroup, but with an unbounded number of defining relations. However, for finite presentation the cancellation laws are essential. The author announces (without detailed proofs) that: there exists a Π_2 that is not isomorphic to any Π; there exists a Π_2 that is not isomorphic to any Π_L or Π_R; there exists a Π_L [Π_R] that is not isomorphic to any Π. The cancellation laws are not even superfluous if identical relations are also admitted. A relation shall be called mixed if it contains variable symbols (for substitution of words) as well as individual words. Defining relations and identical relations are extreme cases of mixed relations. The author proves that there exists a finitely presented Π_2 that is not isomorphic to any Π_L, Π_R or Π with a finite number of mixed relations, and a finitely presented Π_L [Π_R] that is not isomorphic to any Π with a finite number of mixed relations.

K. A. Hirsch (London)

1717:

Saitô, Tôru; and Hori, Shigeo. On semigroups with minimal left ideals and without minimal right ideals. J. Math. Soc. Japan **10** (1958), 64–70.

A semigroup is said (in this review) to be of type $\mathscr{S}$ if it is simple and contains a minimal left ideal but does not contain a minimal right ideal. The authors give the following construction of semigroups of type $\mathscr{S}$. Let $\{K_{\lambda\alpha}:\lambda\in\Lambda,\ \alpha\in A\}$ be a set of disjoint sets. Let $|K_{\lambda\alpha}|=m_\alpha$, where $|X|$ denotes the cardinal of a set X. Let $K(\lambda)=\bigcup\{K_{\lambda\alpha}:\alpha\in A\}$, $K^*(\lambda)=K(\lambda)\cup\{\theta_\lambda\}$. Let $\mathfrak{F}_\mu^\lambda$ consist of all (1, 1)-mappings $f_\mu^\lambda:K^*(\lambda)\to K(\mu)$ such that $f_\mu^\lambda(K_{\lambda\alpha})\subset K_{\mu\alpha}$ and $|K_{\mu\alpha}\setminus f_\mu^\lambda(K_{\lambda\alpha})|=m_\alpha'$. The m_α and m_α' are infinite cardinals. Let $\mathfrak{F}=\bigcup\{\mathfrak{F}(\lambda):\lambda\in\Lambda\}$ where $\mathfrak{F}(\lambda)$ consists of all elements (f_μ^λ) of the product set $\Pi\{\mathfrak{F}_\mu^\lambda:\mu\in\Lambda\}$ such that $f_\mu^\lambda(\theta_\lambda)\in K_{\mu\alpha}$, where α is independent of μ. Define a product in $\mathfrak{F}$ by $(f_\mu^\lambda)(g_\mu^\nu)=(f_\mu^\lambda\circ g_\lambda^\nu)$ where $f_\mu^\lambda\circ g_\lambda^\nu$ is the usual composition of mappings. Then with this product $\mathfrak{F}$ becomes a semigroup of type $\mathscr{S}$. Let S be any semigroup of type $\mathscr{S}$ satisfying the condition: $ax=ay$ for all $a\in S$ implies $x=y$. Then it is shown that S can be embedded in one of the semigroups $\mathfrak{F}$. {A similar embedding theorem for all semigroups of type $\mathscr{S}$ is due to R. Croisot [C. R. Acad. Sci. Paris **239** (1954), 845–847; MR **16**, 215]. The methods used are similar to those of Croisot. No reference is made to Croisot's paper. From Croisot's results it follows that the semigroup $\mathfrak{F}$ in which the authors embed the semigroup S has all the cardinals m_α, m_β' equal.}

G. B. Preston (Shrivenham)

1718:

Steinfeld, O. Über die Quasiideale von Halbgruppen mit eigentlichem Suschkewitsch-Kern. Acta Sci. Math. Szeged **18** (1957), 235–242.

A quasi-ideal of a semigroup S is a subset Q satisfying the relation $SQ\cap QS\subseteq Q$. [This concept was introduced by the author in Publ. Math. Debrecen **4** (1956), 262–275; MR **18**, 790.] In what follows S denotes a semigroup containing a Suschkewitsch kernel K. A quasi-ideal Q of S is called relatively minimal if $K\subset Q$ $(K\neq Q)$ and there does not exist a quasi-ideal Q' with $K\subset Q'\subset Q$ $(K\neq Q'\neq Q)$. The relatively minimal left and right ideals are defined analogously.

The main results of the paper can be formulated as follows: (1) If R [L] are relatively minimal right [left] ideals of S, then either (a) $(RL)^2\subseteq K$ or (b) $RL=R\cap L$ is a relatively minimal quasi-ideal of S. (2) If Q is a relatively minimal quasi-ideal of S, then either (a) $Q^2\subseteq K$ or (b) $Q\setminus K$ is a group. In this second case, we have $Q=K\cup\varepsilon S\varepsilon$ with a suitably chosen idempotent ε.

Št. Schwarz (Bratislava)

1719:

Lefebvre, Pierre. Demi-groupes admettant des complexes nets à droite minimaux. C. R. Acad. Sci. Paris **247** (1958), 393–396.

A subset K of a semigroup D is called a r-net if $xD\cap K\neq\emptyset$ for every $x\in D$. This paper studies semigroups containing minimal r-nets. It is proved that D contains a minimal r-net if and only if it contains a minimal r-ideal. The union of the minimal r-nets of D is the union of the minimal r-ideals of D. If D contains minimal l-nets, the union of the minimal l-nets equals the union of the minimal r-nets. *R. E. Johnson* (Northampton, Mass.)

1720:

Schützenberger, Marcel Paul. Sur les homomorphismes d'un demi-groupe sur un groupe. C. R. Acad. Sci. Paris **246** (1958), 2442–2444.

Let G be a group, let S be a semigroup with identity, and let H be a subset of S. The following two propositions are proven: There exists a homomorphism π of S onto G for which $\pi^{-1}\pi H=H$ if and only if for each $s\in S$ there exists an $n\in Ss\cup sS$ such that $ab\in H$ if and only if $anb\in H$. There exists a homomorphism π of S onto G such that $\pi^{-1}\pi H=H$ and πH consists of a single element if and only if: (1) H intersects all of the ideals of S; (2) $HSHS\cap H\neq\emptyset$; (3) if $xHy\cap H\neq\emptyset$, then $xHy\subset H$ and $x(SHS-H)y\subset S-H$. While proving these propositions the author obtains more general information about the homomorphisms of S onto G. *P. F. Conrad* (New Orleans, La.)

1721:

Schwarz, Štefan. On the structure of the semigroup of measures on a finite semigroup. Czechoslovak Math. J. **7(82)** (1957), 358–373.

Let S be a finite semigroup and $M(S)$ the set of all non-negative measures on S with total measure one. Under convolution $M(S)$ becomes itself a semigroup. This fact has appeared in various contexts [see, e.g., Hewitt and Zuckerman, Duke Math. J., **22** (1955), 595–615; MR **17**, 754]. The present paper examines some of the algebraic properties of $M(S)$ and especially the relation between the right invariant measures on S and the primitive idempotents of $M(S)$. (An idempotent μ is primitive if there is no other idempotent ν such that $\mu*\nu=\nu*\mu=\nu$.) Any right invariant measure is a primitive idempotent, but not conversely. Under the assumption that S has at

least one right invariant measure, the author proves several theorems including the following: The only minimal left ideal of $M(S)$ is its kernel, and this consists precisely of all the primitive idempotents. Further, let N be the kernel of S, and put $R=N-(S-N)S$. Then a primitive idempotent is a right invariant measure if and only if its support is contained in R. It follows from this that the set of all primitive idempotents coincides with the set of all right invariant measures if and only if S is a left simple semigroup.

Many concrete and illustrative examples are included in the paper. *G. Hufford* (Seattle, Wash.)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 1711, 1925, 1926, 1927, 1928, 1929, 2014.

1722:

Berezin, F. A.; and Karpelevič, F. I. Zonal spherical functions and Laplace operators on some symmetric spaces. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 9–12. (Russian)

Gel'fand [Dokl. Akad. Nauk SSSR (N. S.) **70** (1950), 5–8; MR **11**, 498] has formulated the notions of Laplace operators and zonal spherical functions on factor groups of Lie groups with respect to compact subgroups. The present paper computes these operators and functions for three special cases: a complex Grassmann space $M_{n,k}^{+}$, its dual space $M_{n,k}^{-}$, and the "limit" space $M_{n,k}^{0}$ consisting of all complex matrices with k rows and $n-k$ columns. In the case of $M_{n,k}^{+}$ the operator is $j(t)^{-1}P(L_1, \cdots, L_k)j(t)$, where $j(t)=\prod_{p<q}(\sin^2 t_p-\sin^2 t_q)$, $L_p=\frac{1}{2}(\partial^2/\partial t_p^2)+\frac{1}{2}[\cot 2t_p+(n-2k)\cot t_p](\partial/\partial t_p)$, $P(x_1, \cdots, x_k)$ an arbitrary symmetric polynomial of $x_1, \cdots, x_k$. The corresponding function is $d(r_1, \cdots, r_k)\lambda(r_1, \cdots, r_k)/j(t)$, where $d(r_1, \cdots, r_k)=\det\|J_{r_i}^{m}(x_i)\|$, $m=n-2k$, $x_i=\sin^2 t_i$, $J_r^{m}(x)={}_2F_1(m+r+1, -r, m+1; x)$, $\lambda(r_1, \cdots, r_k)=\prod_{s=1}^{k-1} s!(m+s)^{k-s}/\prod_{i<j}(\rho_i-\rho_j)$, $\rho_i=-r_i(m+1+r_i)$, r_i characteristic values. *R. G. Langebartel* (Urbana, Ill.)

1723:

Abe, Eiichi. On simple groups derived from simple Lie rings. Sûgaku **9** (1957/58), 8–10. (Japanese)

Due to Chevalley [Tôhoku Math. J. (2) **7** (1955), 14–66; MR **17**, 457] is a process of deriving some types of simple groups from simple Lie algebras over the complex number field, as subgroups of automorphism groups of Lie algebras over arbitrary fields derived from the original ones over the complex number field. The paper observes the isomorphisms between these simple groups and the so-called classical groups of Dickson and Dieudonné, and considers the simplicity of the said Lie algebras over arbitrary fields. *T. Nakayama* (Nagoya)

1724:

***Dieudonné, Jean. Le calcul différentiel dans les corps de caractéristique $p>0$.** Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 240–252. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

The author describes his differential calculus in power series rings over fields of characteristic p and then proceeds to sketch his theory of formal Lie groups and their Lie hyperalgebras, which is based on this differential calculus. The sketch covers the state of the theory as of 1955 [Comment. Math. Helv. **28** (1954), 87–118; Amer. J. Math. **77** (1955), 218–244; Math. Z. **63** (1955), 53–75; Amer. J. Math. **77** (1955), 429–452; MR **16**, 12, 789; **17**, 174]. It should be noted that, in the meantime, the author has advanced this theory to the point of having obtained decisive results on the isogeny classes of the formal Lie groups. *G. P. Hochschild* (Berkeley, Calif.)

1725:

Mycielski, Jan. Some theorems on connected compact groups. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1023, LXXXV. (Russian summary)

Announcement that (1) a compact group G is connected if and only if every element is a kth power for every integer $k\neq 0$, and (2) all maximal Abelian subgroups of a connected compact group are connected and conjugate.

1726:

Mostert, Paul S. One-parameter transformation groups in the plane. Proc. Amer. Math. Soc. **9** (1958), 462–463.

The author proves the following theorem: Let E be the [Euclidean] plane and R the real line acting in E as a group of transformations without fixed points (i.e., no point is left fixed by all of R). If E/R is Hausdorff, then E is fibred as a direct product of R and a cross sectioning line (i.e., curve). Thus, R is equivalent to a group of translations. That is, there is a simple arc C, $C=\{\varphi(t)| -\infty<t<+\infty\}$ so chosen that the mapping $\Phi: \Phi(a(\varphi(t)))=(a, t)$ from E onto itself is a homeomorphism. Under this homeomorphism, R as a group is equivalent to translation in the horizontal direction. The principal tools used to produce this theorem are the Jordan-Schoenflies theorem and the Brouwer fixed-point theorem. The theorem does not carry over to higher-dimensional Euclidean spaces. *A. Beck* (Madison, Wis.)

TOPOLOGICAL ALGEBRA

1727:

Clifford, A. H. Connected ordered topological semigroups with idempotent endpoints. I. Trans. Amer. Math. Soc. **88** (1958), 80–98.

The author gives a complete classification of compact, connected, linearly ordered topological semigroups with zero and with idempotent endpoints. This extends previous work of Faucett [Proc. Amer. Math. Soc. **6** (1955), 741–747, 748–756; MR **17**, 173], and of Mostert and Shields [Ann. of Math. (2) **65** (1957), 117–143; MR **18**, 809], and partly overlaps recent work of Cohen and Wade [≠1728 below]. *A. Shields* (Ann Arbor, Mich.)

1728:

Cohen, Haskell; and Wade, L. I. Clans with zero on an interval. Trans. Amer. Math. Soc. **88** (1958), 523–535.

The authors give a complete classification of continuous associative multiplications with zero and identity on the unit interval. There is some overlap with recent work of A. H. Clifford [≠1727 above]. They extend earlier work of Faucett, Mostert, and Shields [references in above review]. *A. Shields* (Ann Arbor, Mich.)

1729:

Koch, R. J.; and Wallace, A. D. Admissibility of semigroup structures on continua. Trans. Amer. Math. Soc. **88** (1958), 277–287.

Let S be a continuum. When does S admit a continuous

associative multiplication such that $S\cdot S=S$? This is always possible in two trivial ways: a) $xy=x$ for all $x, y\in S$, b) $xy=y$ for all $x, y\in S$. The authors show that if S is the one sphere, or if S is a metric indecomposable continuum, and if $S\cdot S=S$, then either S is a group or the multiplication is trivial. The problem is open for higher dimensional spheres. They show in general that $S\cdot S=S$ if and only if each dense ideal is connected. They prove a number of other results, especially for the case when S is irreducibly connected between two subsets. Several examples are studied in detail.

A. Shields (Ann Arbor, Mich.)

FUNCTIONS OF REAL VARIABLES

See also 1681, 1906.

1730:

Lipiński, J. S. Sur l'uniformisation des fonctions continues. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 1019–1021, LXXXV. (Russian summary)

Let F be the family of all continuous functions f in $[0, 1]$ such that $0=f(0)\leqq f(x)\leqq f(1)=1$. T. Homma [Kōdai Math. Sem. Rep. **1** (1952), 13–16; MR **14**, 257] proved the theorem: If $f_i\in F$ $(i=1, 2)$ are nowhere constant, then there exist functions $\varphi_i\in F$ $(i=1, 2)$ such that $f_1[\varphi_1(x)]=f_2[\varphi_2(x)]$ for every $x\in[0, 1]$. R. Sikorski and K. Zarankiewicz [Fund. Math. **41** (1955), 339–344; MR **17**, 288] discussed conditions for the existence of such functions $\varphi_i(i=1, 2)$ in a different manner: The existence of φ_1, φ_2 is equivalent to the existence of an arc Γ $(x=\varphi_1(t), y=\varphi_2(t), 0\leqq t\leqq 1)$ joining $p_0=(0, 0)$ and $p_1=(1, 1)$ in the set $A=E_{(x,y)}[f_1(x)=f_2(y)]$. Moreover, for each pair $f_1, f_2\in F$, the points p_0, p_1 lie in the same component A_0 of A. Hence R. Sikorski and K. Zarankiewicz gave conditions for f_1, f_2 to guarantee that every component of the set A be locally connected. Now the author, answering a question of R. Sikorski, proves the theorem: If the functions $f_1, f_2\in F$ have no intervals of constancy, then every component of the set A is locally connected.

A. Rosenthal (Lafayette, Ind.)

1731:

Tanzi Cattabianchi, Luigi. La formula di Taylor-Cauchy. Riv. Mat. Univ. Parma **7** (1956), 359–362.

The author writes the classical formula of Cauchy

$$f(x)=f(x_0)+\frac{f'(x_0+\theta(x-x_0))}{\varphi'(x_0+\theta(x-x_0))}$$

in the form

$$(*)\qquad f(x)=f(x_0)+f_\varphi'(x_0+\theta(x-x_0))\cdot\{\varphi(x)-\varphi(x_0)\}.$$

He then observes that just as (*) with $\varphi(x)=x$ yields Taylor's formula so (*) in general yields a formula which he calls the Taylor-Cauchy formula given by

$$f(x)=f(x_0)+f_\varphi'(x_0)\{\varphi(x)-\varphi(x_0)\}+\cdots+\frac{1}{(n-1)!}f_\varphi^{(n-1)}(x_0)\cdot\{\varphi(x)-\varphi(x_0)\}^{n-1}+T_n(x).$$

In this formula $f_\varphi'(x)=\{\varphi'(x)\}^{-1}f'(x)$, $f_\varphi''=(f''\varphi'-f'\varphi'')\varphi'^{-3}$ etc., and it is assumed that $\varphi'(x)\neq 0$, $a<x<b$.

V. F. Cowling (Lexington, Ky.)

1732:

Mikolás, Miklós. Continuous non-differentiable functions. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 353–367. (Hungarian)

This is essentially the Hungarian version of the author's paper published in Acta Sci. Math. Szeged **17** (1956), 49–62 [MR **18**, 566].

S. Kántor (Debrecen)

1733:

de Rham, Georges. Sur quelques courbes definies par des equations fonctionnelles. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 101–113.

Results of earlier work [École Polytechnique Univ. Lausanne, Centenaire 1853–1953, Lausanne, 1953, pp. 95–97; J. Math. Pures Appl. (9) **35** (1956), 25–42; MR **19**, 842] are here shown to flow from a single source: If $F_a(z)$ $(a=0, 1)$ are contracting transformations of the finite complex plane onto itself (i.e. $|F_a(z_1)-F_a(z_2)|<r|z_1-z_2|$ for a fixed $r<1$, which implies that $F_a(z)$ has a single fixed point f_a) and if $F_1(f_0)=F_0(f_1)$ then the system of functional equations $M(t/2)=F_0(M(t))$, $M((1+t)/2)=F_1(M(t))$, $0\leqq t\leqq 1$, has one and only one bounded solution $M(t)$, and $M(t)$ is continuous.

F. A. Behrend (Melbourne)

1734:

Hájek, O. Note sur la mesurabilité B de la dérivée supérieure. Fund. Math. **44** (1957), 238–240.

The author proves that the upper derivative $\bar{f}$ of a finite real function defined on a 1-dimensional interval I is of the second class of Baire. This is deduced from the facts that, for every real number q, the set of all $x\in I$ with $\bar{f}(x)\geqq q$ is a countable intersection of unions of non-degenerate intervals and that each such union is an F_σ-set whose complement to I is the difference of a closed and a countable set.

H. M. Schaerf (St. Louis, Mo.)

1735:

Codyks, V. M. On sets of points where the derivative is equal to $+\infty$ or $-\infty$ respectively. Mat. Sb. N.S. **43(85)** (1957), 429–450. (Russian)

In order that E^1 and E^2 are the sets of all points where the derivative of some function $f(x)$ of a real variable, finite at every x, exists and is equal to $+\infty$ and $-\infty$, respectively, it is necessary and sufficient that (i) E^1 and E^2 are $F_{\sigma\delta}$ of measure 0, (ii) there exist H^1, H^2 of type F_σ such that $E^1\subset H^1$, $E^2\subset H^2$ and $H^1\cap H^2=0$. [This strengthens a result by Landis, Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 202–204; MR **17**, 1190.]

F. A. Behrend (Melbourne)

1736:

Jůza, Miloslav. Sur certaines classes de fonctions continues. Časopis Pěst. Mat. **83** (1958), 83–90. (Czech. Russian and French summaries)

Let $0<\sigma<\tau\leqq 1$. Auerbach and Banach proved [Studia Math. **3** (1931), 180–184] the existence of a function $f(x)$ satisfying (i) a Lipschitz condition of order σ, (ii) $\limsup_{x\to x_0}|f(x)-f(x_0)|/|x-x_0|^\tau=\infty$ for all x_0. The author gives an explicit construction of such a function. It follows that there exists, for each natural number m, a continuous function $f(x)$ such that (i) for any non-constant polynomial P of degree less than m, $P(f(x))$ nowhere possesses a finite derivative, (ii) there exists a polynomial Q of degree m such that $dQ(f(x))/dx=0$ for one point x at least.

F. A. Behrend (Melbourne)

1737:

Kassimatis, C. Functions which have generalized Riemann derivatives. Canad. J. Math. **10** (1958), 413–420.

The nth generalized Riemann derivative of a measurable function f at x, denoted by $D^n f(x)$, is

$$\lim_{h\to 0}(2h)^{-n}\,\Delta_n(x, 2h; f)$$

(if this limit exists and is finite at x), where

$$\Delta_n(x, 2h; f)=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(x+2jh-nh)$$

$(h>0;\ n=1, 2, 3, \cdots)$.

Let $F(x)$, $G(x)$ be defined in $[a, b]$ such that $F-G$ is continuous in $[a, b]$. If $D^2(F-G)=0$ for all $x\in(a, b)$, then, as the author proves, $F-G$ is linear in $[a, b]$. This result, under the stronger assumption that both functions F and G are continuous, was already proved (but in quite a different way) by R. D. James [Canad. J. Math. **2** (1950), 297–306; MR **12**, 94] and by R. L. Jeffery [Trigonometric Series, Univ. of Toronto Press, 1956, § 14]. The author then discusses the cases where $D^n(F-G)=0$ for all $x\in(a, b)$ with $n>2$. In order to obtain the analogous result here that $F-G$ is a polynomial of degree at most $n-1$, additional conditions must be imposed on $F-G$ (already for $n=3$). The author establishes such conditions for $n>2$. His method is based on a generalization of a theorem which for the case $n=2$ was given by A. Denjoy [Leçons sur le calcul des coefficients d'une série trigonométrique, Tome I, Gauthier-Villars, Paris, 1941; MR **8**, 260; pp. 18–19].

A. Rosenthal (Lafayette, Ind.)

1738:

Aczél, J. Bemerkungen zu einigen Methoden bezüglich der elementaren Lösung von Extremumsaufgaben mittels elementaren Ungleichungen. Acta Univ. Debrecen **3** (1956), no. 2, 23–40. (Hungarian. German summary)

Elementary (non-calculus) inequalities used, and proved: $\sum q_i a_i\geqq\prod a_i^{q_i}(\sum q_i=1, q_i\geqq 0, a_i\geqq 0)$ and its subcases $(1+t)^s\geqq 1+ts$ and det (pos. binary quadratic)$\geqq 0$. The author follows Perelman [Entertaining geometry and Entertaining algebra, Moscow-Leningrad, 1950], Natanson [Simple maximum and minimum problems, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1951], and Korovkin [Inequalities, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952; MR **14**, 24]. The question is raised whether elementary extremal problems exist that do not admit an elementary solution.

T. S. Motzkin (Los Angeles, Calif.)

1739:

Whiteley, J. N. Some inequalities concerning symmetric forms. Mathematika **5** (1958), 49–57.

Let $a=(a_1, \cdots, a_m)$, $b=(b_1, \cdots, b_m)$ with each $a_j\geqq 0$, $b_j\geqq 0$. For a real parameter $s\neq 0$, let $T_n(a)$ be defined by the generating functions

$$\sum_{n=0}^{\infty}T_n(a)x^n=\prod_{j=1}^{m}(1+a_jx)^s, \text{ if } s>0;$$

$$\sum_{n=0}^{\infty}T_n(a)x^n=\prod_{j=1}^{m}(1-a_jx)^s, \text{ if } s<0.$$

The following theorem is proved by induction on m and n. If $s<0$, then

$$(T_n(a+b))^{1/n}\leqq(T_n(a))^{1/n}+(T_n(b))^{1/n}\ (n\geqq 1).$$

If $s>0$, the reversed inequality holds, provided that $n<s+1$ in case s is not an integer. When $s=1$, this result becomes an inequality obtained by H. F. Bohnenblust [unpublished] and by M. Marcus and L. Lopes [Canad. J. Math. **9** (1957), 305–312; MR **18**, 877]. The classical Minkowski's inequality may be regarded as the limiting case $s\to 0$ from the negative side.

Ky Fan (Notre Dame, Ind.)

1740:

Berruti Onesti, Natalia. Sopra un'estensione del lemma di Gronwall. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **91** (1957), 643–649.

Let $I=[x_0, x_0+h]$, $J=[y_0, y_0+k]$, $R=I\times J$, and let $I_x=[x_0, x]$, $J_y=[y_0, y]$, $R_{xy}=I_x\times J_y$ for any $(x, y)\in R$. Suppose $M(x)$, $N(y)$, $A(x, y)$, $z(x, y)$ are functions which are non-negative, continuous a.e., and Lebesgue integrable on I, J, R, respectively, such that $M(x)N(y)z(x, y)$ is Lebesgue integrable on R. The author proves the following generalization of Gronwall's lemma. If in R a.e.

$$z(x, y)\leqq\iint_{R_{xy}}[M(u)N(v)z(u, v)+A(u, v)]dudv+B,$$

where B is a non-negative constant, then in R a.e.

$$z(x, y)\leqq\Big[\iint_R A(u, v)dudv+B\Big]\exp\Big(\int_{x_0}^{x}M(t)dt+\int_{y_0}^{y}N(t)dt\Big).$$

H. A. Antosiewicz (Los Angeles, Calif.)

1741:

Moldovan, Elena. Sur une généralisation de la notion de convexité. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Şti. Ser. I. **6** (1955), no. 3–4, 65–73. (Romanian. Russian and French summaries)

In defining generalized convex functions, one considers a basic n-parameter family $\mathfrak{F}$ of real-valued functions $y=F(x; a_1, \cdots, a_n)$ defined and continuous on an interval $a\leqq x\leqq b$ and such that, for any set of n distinct points $x_1, \cdots, x_n$ of $[a, b]$ and any set of n real values $y_1, \cdots, y_n$, the system of equations $F(x_i; a_1, \cdots, a_n)=y_i$, $i=1, \cdots, n$, has exactly one solution in $\mathfrak{F}$. The author denotes the solution by $L(x_1, \cdots, x_n; y|x)$ and establishes the following concerning the family $\mathfrak{F}$. Let E_m be a set of m distinct points x_j of the interval $[a, b]$, $a\leqq x_1<\cdots<x_m\leqq b$, $m\geqq n+1$, let Y be an arbitrary set of real values $y_1, \cdots, y_m$, let the indices $i_1, \cdots, i_n$ satisfy $1\leqq i_1, <\cdots<i_n\leqq m$, and let x_0 satisfy $x_{i_n}<x_0\leqq b$; then the value $L(x_{i_1}, x_{i_2}, \cdots, x_{i_n}; y|x_0)$ satisfies the inequalities

$$\min_{j=i_1, i_1+1, \cdots, i_n-n+1}L(x_j, x_{j+1}, \cdots, x_{j+n-1}; y|x_0)$$

$$\leqq L(x_{i_1}, x_{i_2}, \cdots, x_{i_n}; y|x_0)$$

$$\leqq\max_{j=i_1, i_1+1, \cdots, i_n-n+1}L(x_j, x_{j+1}, \cdots, x_{j+n-1}; y|x_0).$$

The function $f(x)$, defined on a set E_m of m distinct points x_j of interval $[a, b]$, $a\leqq x_1<\cdots<x_m\leqq b$, is said to be convex (nonconcave, nonconvex, concave) on E_m relative to the family $\mathfrak{F}$ provided that for every set of $n+1$ points $x_{i_1}<\cdots<x_{i_{n+1}}$ of E_m we have

$$L(x_{i_1}, \cdots, x_{i_n}; f|x_{i_{n+1}})<(\leqq, \geqq, >)f(x_{i_{n+1}}).$$

It follows from the result of the preceding paragraph that the function $f(x)$ is convex (nonconcave, nonconvex, concave) relative to the family $\mathfrak{F}$ on E_m if and only if $f(x)$ is convex (nonconcave, nonconvex, concave) relative to the family $\mathfrak{F}$ on every set of $n+1$ consecutive points $x_j, \cdots, x_{j+n}$, $j=1, \cdots, m-n$, of the set E_m. Applications are given to the discussion of the continuity of generalized convex functions.

E. F. Beckenbach (Los Angeles, Calif.)

MEASURE AND INTEGRATION

See also 1721, 1959, 2041.

1742:

Denjoy, Arnaud. Sur le théorème de Vitali. Rev. Math. Pures Appl. 2 (1957), 161–166.

The strong Vitali theorem for the plane states that if every point M of a set H is the center of a sequence of squares $c_n(M)$ whose sides approach zero with n, then there exists a sequence of disjoint $c_n(M)$ covering H except for a set of zero measure, the sum of whose areas exceeds the measure of H by an arbitrary ε at most. The squares can be replaced by rectangles if the ratio of the sides lies between k and $1/k$, k fixed. The note gives an example in which the theorem still holds although no k exists limiting the ratio of the sides of the covering rectangles. The weak Vitali theorem has as conclusion that for every $\varepsilon>0$, there exists a denumerable set of the $c_n(M)$ such that the sum of the areas of $c_n(M)$ exceeds the measure of H by ε at most. The note gives an example of a covering of the open interval $0<x<1$ by circles and half circles, whose centers are not on the interval (0, 1), where the theorem holds in the strong sense but not in the weak.

T. H. Hildebrandt (Ann Arbor, Mich.)

1743:

Denjoy, Arnaud. Le théorème de Vitali. Bull. Math. Soc. Math. Phys. R. P. Roumaine (N.S.) **1(49)** (1957), 11–15.

This is a brief resumé of results published in a series of notes in C. R. Acad. Sci. Paris, **231** (1950), 560–562, 600–601, 737–740, 1013–1015; **232** (1951), 195–197, and in greater detail in Amer. J. Math. **73** (1951), 314–356 [MR 12, 246, 324, 398, 685].

T. H. Hildebrandt (Ann Arbor, Mich.)

1744:

Poprużenko, J. Sur certaines représentations des fonctions d'ensemble à variation bornée. I. Colloq. Math. **5** (1957), 43–50.

$\mathscr{M}$: ensemble abstrait de puissance m. $\mathfrak{M}$: σ-corps de sous-ensembles E de $\mathscr{M}$. $\mathfrak{H}$: famille des ensembles H de $\mathfrak{M}$ tels que $(Z \subset H) \Rightarrow (Z \in \mathfrak{M})$. $\mathfrak{B}$: σ-corps inclus dans $\mathfrak{M}$ tel que, pour tout ensemble E de $\mathfrak{M}$, il existe B dans $\mathfrak{B}$ vérifiant $B \subset E$ et $E-B \in \mathfrak{H}$. Deux ensembles E_1 et E_2 sont dits "séparables ($\mathfrak{B}$)" s'il existe B dans $\mathfrak{B}$ tel que $E_1 \subset B$ et $E_2 \subset \mathscr{M}-B$.

Hypothèse $P(\mathfrak{m})$: il n'existe aucun aleph inaccessible $\leqq \mathfrak{m}$. F: fonction réelle, définie sur $\mathfrak{M}$, de variation totale V bornée, additive sur tout couple d'ensembles E_1 et E_2 de $\mathfrak{M}$ séparables ($\mathfrak{B}$) et σ-additive sur $\mathfrak{B}$. Théorème 1. — F est représentable de manière unique sous la forme $\varphi+G$ où $\varphi(E)=\varphi(E \cdot H_0)$ pour un certain ensemble $H_0 \in \mathfrak{H}$ et G est une fonction σ-additive (sur $\mathfrak{M}$) nulle sur $\mathfrak{H}$. Théorème 2. — Si $P(\mathfrak{m})$ est vérifiée et si F s'annule sur les ensembles uniponctuels de $\mathfrak{M}$, tout $E \in \mathfrak{M}$ admet une décomposition en ensembles X_i ($i=0, 1, 2, \cdots$) de $\mathfrak{M}$ deux à deux disjoints telle que $((Z \subset X_i) \& (Z \in \mathfrak{M})) \to (\varphi(Z)=0)$. Dans les Théorèmes 3 et 4, μ désigne une mesure finie sur $\mathfrak{M}$, $\mathfrak{H}$ est la famille des ensembles de mesure nulle; il s'agit principalement de la représentation de F comme μ-intégrale.

Indications sur les démonstrations. — (Th. 1) H_0 est un ensemble de $\mathfrak{H} \cdot \mathfrak{B}$ tel que $V(H_0)=\sup V(H)$ pour $H \in \mathfrak{H}$, $\varphi(E)=F(E \cdot H_0)$. (Th. 2) L'outil principal est le "Théorème de Recouvrement de la Théorie Générale des Ensembles" de W. Sierpiński, Fund. Math. **20** (1933), 214–220. {Remarques du rapporteur — (i) La possibilité de trouver un ensemble maximal H_0 dans $\mathfrak{B}$ semble dépendre de la présence de $\mathscr{M}$ dans $\mathfrak{M}$. (ii) La décomposition d'une mesure sur une σ-algèbre booléenne $\mathfrak{M}$, relativement à un σ-idéal $\mathfrak{H}$ de $\mathfrak{M}$, est donnée dans M. Nakamura et G. Sunouchi, Proc. Imp. Acad. Tokyo **18** (1942), 333–335 [MR 8, 256]}. *Chr. Y. Pauc* (Nantes)

1745:

Lamperti, John. On flows of measure-preserving transformations. Michigan Math. J. **4** (1957), 161–164.

Let T be an invertible ergodic measure preserving transformation of the unit interval onto itself with a pure point spectrum (discrete spectrum). Such a transformation is called by the author a d.s. transformation. The author discusses the questions of the existence of nth roots of d.s. transformations and the embeddability of d.s. transformations in flows.

The author proves th. 1: A d.s. transformation T has an nth root if and only if its spectrum contains no nth root of unity other than 1. He also gives a necessary and sufficient condition for the embeddability of a d.s. transformation in a flow. The above results have a number of corollaries among which is the following: There exist d.s. transformations which have roots of all orders but are not embeddable in a flow. Th. 1 is a generalization from $n=2$ of a theorem of Halmos [Amer. J. Math. **64** (1942), 153–166; MR 3, 211]. *Y. N. Dowker* (London)

1746:

Mišík, L. Der Mittelwertsatz für additive Intervallfunktionen. Fund. Math. **45** (1957), 64–70.

φ: real-valued additive function defined on the subintervals I of a fundamental interval I_0 in the n-dimensional euclidean space E_n. c: number. x, y, z: points of I_0. $D\varphi(x)$, $D_m\varphi(x)$, $\varphi'(x)$: ordinary, cube, strong derivative (according to Saks) of φ at x, if it exists.

Theorem (Darboux property of $D_m\varphi$): If $D_m\varphi$ exists and is finite on I_0, if $D_m\varphi(y)>c$ and $D_m\varphi(z)<c$, then there exists a point ξ interior to I_0 such that $D_m\varphi(\xi)=c$. Mean Value Theorems: If $D\varphi$ [φ'] exists and is finite on I_0, then any subinterval I of I_0 contains an interior point ξ such that $\varphi(I)=D\varphi(\xi) \cdot |I|$ [$\varphi'(\xi) \cdot |I|$].

Hints as to proofs: (a) The following proposition is assumed to be known: if the lower cube derivate $\underline{D}_m\varphi(x)$ is $\geqq 0$ [>0] at each point of I, then $\varphi(I) \geqq 0$ [>0]. (b) Assuming $\underline{D}_m\varphi(y)>0$ and $\overline{D}_m\varphi(z)<0$, a sequence $K_1, \cdots, K_i, \cdots$ of cubes is constructed, having the following properties: $K_{i+1} \subseteq \operatorname{Int} K_i$, the diameter of K_i tends to 0 when $i \to \infty$, and $\varphi(K_i)$ is alternately positive or negative. Hence at the point ξ on which K_i contracts, $\underline{D}_m\varphi(x)$ is $\leqq 0$, $\overline{D}_m\varphi(x)$ is $\geqq 0$. *Chr. Y. Pauc* (Nantes)

1747:

Foiaş, Ciprian. De l'intégrabilité Stieltjes-Riemann par rapport à une fonction qui n'est pas à variation bornée. Com. Acad. R. P. Romîne **7** (1957), 835–837. (Romanian. Russian and French summaries)

The author shows that the set of continuous functions on a given interval which are Riemann-Stieltjes integrable with respect to a given function not of bounded variation is of the first category in the space of continuous functions on the interval in question.

I. I. Hirschman, Jr. (St. Louis, Mo.)

1748:

Snow, D. O. On integration of vector-valued functions. Canad. J. Math. **10** (1958), 399–412.

The author considers functions defined on subsets of

Euclidean n-space, with values in a Banach space X. (For simplicity of notation the case of subsets of a bounded linear interval $[a, b]$ is taken.) Corresponding to a function $x(\alpha)$ defined on a closed subset P of $a \leqq \alpha \leqq b$, $(G)\int_P x(\alpha)d\alpha$ is defined as the (strong) limit (when it exists) of sums of the form $\sum_t x(\xi_t)|P \cap \Delta\alpha_t|$, where the $\Delta\alpha_t$ are nonoverlapping subintervals, filling up $[a, b]$. When $x(\alpha)$ is defined on a measurable set E, it is said to be P_ε-measurable on E if it is G-integrable over P for closed subsets P of E with $|E-P|$ arbitrarily small. The integral $(G^*)\int_E x(\alpha)d\alpha$ of a P_ε-measurable function x is then defined as the limit (when existent) of $(G)\int_P x(\alpha)d\alpha$ as $|E-P|$ tends to zero, where P is a closed subset of E. For the case being considered, the author shows that the G^*-integral is equivalent to the Birkhoff integral.

L. M. Graves (Chicago, Ill.)

1749:

Mařík, Jan. Eine Bemerkung über die Länge einer Jordanschen Kurve. Časopis Pěst. Mat. **83** (1958), 91–96. (Czech. Russian and German summaries)

For every measurable set A of the complex z-plane, $z=x_1+ix_2$, let P_A [B_A] be the class of all real [complex] polynomials $p(x_1, x_2)$ such that $|p(x_1, x_2)| \leqq 1$ for all $x_1+ix_2 \in A$. Let $\|A\|_k = \mathrm{Sup}(A)\int (\partial p/\partial x_k)d\mu$, $k=1, 2$, where μ is the Lebesgue measure in the z-plane, and Sup is taken in P_A, and $\|A\| = \mathrm{Sup}\,(A)\int \mathrm{div}\, p\, d\mu$, where Sup is taken in B_A. Let $C: z=f(t)$, $a \leqq t \leqq b$, $f=f_1+if_2$, be any closed simple curve in the z-plane, G the interior of C, d the Jordan length of C, $0<d \leqq +\infty$, v_k the total variation of f_k, $k=1, 2$. The author proves that for any measurable set A, $GCA C\bar{G}$, we have $v_1=\|A\|_2$, $v_2=\|A\|_1$, $d=\|A\|$.

L. Cesari (Baltimore, Md.)

1750:

Cecconi, Jaures. Rettifica alla nota "La disuguaglianza di Cavalieri per la k-area secondo Lebesgue in un n-spazio". Ann. Mat. Pura Appl. (4) **44** (1957), 171.

The author rectifies an oversight in his definition of the $(k-1)$ areas of the sections of a k-surface given in his previous paper [same Ann. (4) **42** (1956), 189–204; MR **19**, 735]. The same arguments of his previous paper apply without modification.

L. Cesari (Baltimore, Md.)

1751:

Federer, Herbert. A note on the Gauss-Green theorem. Proc. Amer. Math. Soc. **9** (1958), 447–451.

Let $\nu(A, x)$ denote the exterior unit normal to a set A in euclidean n-space at the point x, as previously defined by the author; and N the set of points at which the exterior normal exists. Suppose that the gradient (in Schwartz distribution theory sense) of the characteristic function of A is a finite vector-valued measure Φ. De Giorgi [Ricerche Mat. **4** (1955), 95–113; MR **17**, 596] showed that there is a Borel set $F \subset N$ such that, for every Borel set S:

$$(*) \qquad -\Phi(S) = \int_{S \cap F} \nu(A, x) dH_n{}^{n-1}x,$$

where $H_n{}^{n-1}$ is Hausdorff $(n-1)$-measure. It is now shown that $H_n{}^{n-1}(N-F)=0$, which implies that F can be replaced by N in (*). Since any set A has the property in question if $H_n{}^{n-1}$ (Boundary A) $<\infty$, this yields in particular a new proof of the author's version of the Gauss-Green theorem [Trans. Amer. Math. Soc. **58** (1945), 44–76; **59** (1946), 441–466; **62** (1947), 114–192; MR **7**, 199, 422; **9**, 231].

W. H. Fleming (Providence, R.I.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 1789, 1854, 1886, 1909, 1910, 1911, 1998, 2203.

1752:

Chuang, Chi-tai. Un théorème général sur les fonctions holomorphes dans le cercle unité et ses applications. Sci. Record (N.S.) **1** (1957), no. 1, 37–40.

Using a theorem that he has previously established [Sci. Record **5** (1952), 1–9; MR **15**, 295] concerning increasing convex functions, the author proves a result, for functions $f(z)$ analytic in the unit circle $|z|<1$, concerning the maximum absolute value $M(r, f)$ of $|f(z)|$ on $|z|=r<1$; he then indicates some consequences of which the following is typical. Let $F(z)$ be analytic for $|z|<1$, with $F(z) \neq 0$, let C, λ, K, and p be constants, with $C>0$, $\lambda>0$, $K>1$, and p a positive integer, and suppose there is a value r_0, $0<r_0<1$, such that

$$M(r_0, \mathrm{Log}\, F) \geqq C(\log K + p + |\mathrm{Log}\, F(0)|)e^{-\lambda/\log r_0};$$

then there exist an integer $q \geqq 0$ and a function $\phi(u)$, analytic and schlicht in the rectangle R: $-\log K < \mathrm{Re}(u) < \log K$, $2\pi q < \mathrm{Im}(u) < 2\pi q + 2(p+1)\pi$, for which, in R, we have $|\phi(u)|<1$ and $F[\phi(u)]=e^u$.

E. F. Beckenbach (Los Angeles, Calif.)

1753:

Lavrentieff, M. A. Sur la théorie des représentations quasi-conformes. Ann. Acad. Sci. Fenn. Ser. A. I, no. **250/18** (1958), 8 pp.

This is a short exposition, without proofs, of some of the author's results about mappings by a pair of functions of two real variables connected by an elliptic system of two nonlinear partial differential equations of first order. Also extensions to three dimensions are discussed, including the concept of a harmonic mapping. This is a mapping of a domain homeomorphic to a ball onto a ball which preserves level surfaces of Green's functions and their orthogonal trajectories, and is conformal at the point at which the Green's function is singular. Such mappings have some properties in common with conformal mappings.

L. Bers (New York, N.Y.)

1754:

Roşculet, Marcel N. Sur les dérivées partielles polydimensionelles et les fonctions monogènes. Bul. Inst. Politehn. Bucureşti **18** (1956), no. 1–2, 11–16. (Romanian. Russian and French summaries)

"L'auteur montre que les fonctions monogènes dans l'algèbre définie par la relation $\theta^3=1$ ont la dérivée tridimensionnelle indépendante par rapport au trièdre considéré et réciproquement." (Author's summary.)

J. A. Ward (Holloman A.F.B., N.M.)

1755:

Nisigaki, Hisami. A theory of functions of a ternary complex variable. II. Mem. School Sci. Engrg. Waseda Univ., Tokyo no. **19** (1956), 82–102.

This paper consists of Chapters II, III, and IV of the treatment begun in the author's previous paper [same Mem., no. **19** (1954), 110–122; MR **17**, 28], in which the notations have been defined.

In Chapter II (Absolute value, modulus, and amplitudes) the author defines absolute value of an element in terms of its components and the type of algebra. This definition is used to define convergent series, modulus, amplitude, and logarithm.

In Chapter III (Functions of a ternary complex variable) derivative is defined as a limit of a quotient. Generalized Cauchy-Riemann first order homogeneous

differential equations are developed with constant coefficients.

In Chapter IV (Integrals of the differentiable functions) the author defines integration and obtains a generalization of Cauchy's Integral Formula.

J. A. Ward (Holloman A.F.B., N.M.)

1756:

Takasu, Tsurusaburo. Some classes of function-theoretical solutions of the Laplacian and the wave equation. Yokohama Math. J. **4** (1956), 65–79.

This paper is based on the author's previous definition of holomorphic function in 3-dimensional hypercomplex function theory. See H. Nisigaki, Mem. School Sci. Engrg. Waseda Univ., Tokyo, no. **18** (1954), 110–122 [MR **17**, 28], and #1755 above; H. Nisigaki and T. Takasu, Yokohama Math. J. **3** (1955), 53–126 [MR **18**, 725.]

The author shows how to choose the multiplication constants of the system in such a way that each component of any twice differentiable holomorphic (in the author's sense) function will satisfy the 3-dimensional Laplacian. He does the same for the 2-dimensional wave equation. *J. A. Ward* (Holloman A.F.B., N.M.)

1757:

Makai, E. On a maximum problem. Acta Math. Acad. Sci. Hungar. **9** (1958), 105–110.

Put $P(z)=\prod_1^n (1+u_\iota z)$, $(P(z))^{-1}=\sum_0^\infty b_\iota z^\iota$, $h_m(z)=(-1)^{m+1}(1-P(z)\sum_0^m b_\iota z^\iota)$. The author wants to estimate the maximum K_{mn} of $\int_0^{2\pi}|h_m(e^{i\theta})|^2 d\theta$ under the condition that $|u_1|\leqq 1, \cdots, |u_n|\leqq 1$. He shows that the coefficients of the polynomial $h_m(z)$ are symmetrical functions of $u_1, \cdots, u_n$ with non-negative coefficients. It follows that the maximal value of the integral is attained if $u_1=\cdots=u_n=1$. In this case the coefficients of $h_m(z)$ can be evaluated explicitly, and so an elementary expression for K_{mn} can be derived. Some simpler upper and lower bounds for K_{mn} are given. *N. G. de Bruijn* (Amsterdam)

1758:

Rooney, P. G. On some properties of functions regular in the unit circle. Canad. Math. Bull. **1** (1958), 25–29.

An analytic function $f(z)$ regular in unit circle is said to belong to the space $H_{\lambda,p}$, $\lambda>0$, if $M_{\lambda,p}(f)$ is finite, where $M_{\lambda,p}(f)=\int_0^1 (1-r^2)^{q\lambda-1}[M_p(f,r)]^q r dr$, $1<p\leqq\infty$, $p^{-1}+q^{-1}=1$, while $M_{\lambda,1}(f)=\sup_{0\leq r<1}(1-r^2)^\lambda M_1(f,r)$; here $M_p(f,r)$ is the mean modulus of order p of $f(z)$. $H_{0,p}$ is the well known space H_p. This paper presents two theorems which generalize to $H_{\lambda,p}$ well-known results for H_p.

Theorem 1: If for some p, $1\leqq p\leqq 2$, and some $\lambda\geqq 0$, $f\in H_{\lambda,p}$, where $f(z)=\sum_{n=0}^\infty a_n z^n$, then $\{n^{-\lambda}a_n, n=1, 2, \cdots\}\in l_q$, where $p^{-1}+q^{-1}=1$. Since $M_{\lambda,p}(f)<\infty$ it follows that $M_p(f,r)<\infty$, hence $f\in L_p$. Upon employing the Hausdorff-Young theorem, forming the integral representation for $M_{\lambda,p}(f)$ and integrating, the theorem follows. Theorem 2: If for some P, $1\leqq p\leqq 2$, and some $\lambda\geqq 0$, $\{n^{-\lambda}a_n, n=1, 2, \cdots\}\in l_p$, then $f\in H_{\lambda,q}$, where $p^{-1}+q^{-1}=1$.

The proof of theorem 2 uses the methods of theorem 1 employed in different order.

W. C. Royster (Lexington, Ky.)

1759:

Lavrentieff, M. M. Uniqueness and stability of analytic functions. Ann. Acad. Sci. Fenn. Ser. A. I. no. **250/19** (1958), 6 pp.

Let $f(z)$ be analytic in the unit circle D: $|z|<1$, let A be a set of points $a_1, a_2, \cdots, a_n, \cdots$ such that $|a_n|<|a_{n+1}|$, $|a_n|\leqq|a|<1$, let $|f(z)|\leqq 1$ in D, and let $|f(z)|\leqq\varepsilon$ in A. Theorem 1, given without proof, states that for each z_0 in D, $|f(z_0)|\leqq\beta^N$ where $\beta=\beta(a,z_0)<1$, $N=N(A,a,z_0,\varepsilon)\to\infty$ as $\varepsilon\to 0$. Explicit expressions are given for β and N. The author then sketches the proof of an analogous result, Theorem 2, for a function $f(z_1,z_2)$ which is analytic in the unit sphere D: $|z_1|^2+|z_2|^2\leqq 1$, $z_1=x+iu$, $z_2=y+iv$, and which is real on the real plane. Finally a plane set A is said to be a set of uniqueness if it has the following property: Given any real function $f(x,y)$ which is analytic in x and y in a domain which contains A and whose boundary lies at a positive distance from A; if f vanishes on A, then f vanishes identically. Theorem 2 yields conditions which are sufficient to guarantee that certain sets are sets of uniqueness. *F. W. Gehring* (Helsinki)

1760:

Davis, Philip; and Pollak, Henry. On the analytic continuation of mapping functions. Trans. Amer. Math. Soc. **87** (1958), 198–225.

Let B be a finite region in the complex plane whose boundary b consists of $N+1$ analytic curves b_ν. Let $g(w)$ be analytic in B and continuous in $B+b$. The linear transform $f(z)=T[g(w)]=\pi^{-1}\iint_B \overline{g(w)}/(z-w)dA_w$ is defined; it is an analytic function in the complement $\tilde{B}$ of B and closely related to the Hilbert transform of $g(w)$ [see Bergman and Schiffer, Compositio Math. **8** (1951), 205–249; MR **12**, 602]. The inversion formula $\overline{g(w)}=(2i)^{-1}\int_b K(z,\bar{w})f(z)dz$ is proved where K is the Bergman kernel of B. The necessary and sufficient condition that $g(w)$ can be continued analytically from B through b into $\tilde{B}$ is that its transform $f(z)$ can be continued analytically from $\tilde{B}$ through b into B. By Green's theorem, the T-transformation may be expressed as a contour integral over b; expressing $\bar{w}=S_\nu(w)$, where S_ν is analytic on b_ν, the method of contour integrals of analytic functions may be brought into play. $S_\nu(w)$ is called the Schwarz function of the curve b_ν. It is easily expressed by means of the analytic function $m_\nu(\zeta)$ which maps the unit circle onto the interior B_ν of b_ν. Let us assume that $g(w)$ possesses a single-valued integral $h(w)$ in B and let $\overline{h(\bar{z})}=\bar{h}(z)$. We designate by $f_\nu(z)$ the component of $f(z)$ defined in B_ν. Then $f_\nu(z)$ and $\bar{h}[S_\nu(z)]$ are shown to have the same singularity character in B, that is, $f_\nu(z)-\bar{h}[S_\nu(z)]$ can be continued analytically over the whole domain B. Next, let B be simply-connected, contain the point $z=0$ and let $m(\zeta)$ map the unit circle onto B such that $m(0)=0$. The following results are typical: a) The condition

$$\lim_{n\to\infty}\left[\iint_B z^n\overline{g(z)}dA\right]^{1/n}=0$$

is necessary and sufficient in order that $h[m(\zeta)]$ be an entire function; b) the function $h[m(\zeta)]$ is a polynomial of degree n if and only if $\iint_B z^p\overline{g(z)}dA=0$ for $p\geqq n$. Finally, the analytic character of the Schwarz function is discussed. *M. M. Schiffer* (Stanford, Calif.)

1761:

Gheorghiu, Şerban. On some integrals. Gaz. Mat. Fiz. Ser. A. **2** (1957), 70–75. (Romanian)

The integrals $I_n=\int_0^\infty \varphi(x)[(\ln x+2n\pi i)^2+\pi^2]^{-1}dx$ and $I_n'=\int_0^\infty \varphi(x)[\ln^2 x+(2n+1)^2\pi^2]^{-1}dx$ are computed by the method of residues. Here $\varphi(x)$ stands for a rational function, regular at the origin and with $\lim_{|x|\to\infty}|x\varphi(x)|=0$.

E. Grosswald (Philadelphia, Pa.)

1762:

Havin, V. P. On analytic functions representable by an integral of Cauchy-Stieltjes type. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. 13 (1958), no. 1, 66–79. (Russian. English summary)

A number of theorems are stated and proved giving necessary and/or sufficient conditions for representing a function $u(z)$, analytic on the complement G of an infinite, closed, proper subset E of the extended complex plane, by an integral of Cauchy-Stieltjes type:

$$(1) \qquad u(z)=\int_E \frac{dg(e)}{\zeta-z}, \quad z\in G,$$

where $g(e)$ is a measure defined on subsets of E.

In addition the following theorem is stated and proved: Let L be an analytic Jordan curve, D_L its interior. Then if in the interior of the unit circle $u(z)$ is representable as a Cauchy-Stieltjes integral in the form

$$(2) \qquad u(z)=\int_0^{2\pi} \frac{dg(\theta)}{e^{i\theta}-z},$$

then the function $u[\psi(w)]$, $w\in D_L$, where $z=\psi(w)$ effects a conformal mapping of D_L onto $|z|<1$, is again representable in the form (2), namely

$$(3) \qquad u[\psi(w)]=\int_L \frac{d\Phi(\tau)}{\tau-w}.$$

J. F. Heyda (Cincinnati, Ohio)

1763:

Mel'nik, I. M. An exceptional case in the Riemann boundary problem. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 24 (1957), 149–162. (Russian)

The Riemann problem is to find a piecewise holomorphic function $\Phi(z)$ bounded at infinity and satisfying the condition $\Phi^+(t)=G(t)\Phi^-(t)+g(t)$ on a contour L, which consists of a finite number of closed or open arcs bounding a region $(+)$ and its complement $(-)$ in the complex plane; here $G(t)$ and $g(t)$ are given functions on L satisfying a Hölder condition, where everywhere on L the order of $G(t)$ is zero. The "exceptional case" of this paper is that in which the arcs of L are all open and the order of $G(t)$ differs from zero at the ends of these arcs.

Expressing $G(t)$ in the form $G(t)=(t-c_k)^{r_k}G^*(t)$, where c_k is either end point of the kth arc, r_k is some real number, and $G^*(t)$ is of order zero near c_k, the author considers the Cauchy integral

$$\Gamma(z)=\frac{1}{2\pi i}\int_L \frac{\ln G(t)}{\tau-z}\,d\tau.$$

A preliminary study of integrals of this type has given him a computational formula for $\Gamma(z)$, and by tracing the course of the parameter t, he is able to prove that the solution of the homogeneous (i.e. $g(t)\equiv 0$) problem is $X(z)=\Pi(z)e^{\Gamma(z)}$, where $\Pi(z)$ is a product of negative powers of $(z-c_k)$, the exponents depending on r_k and $\arg(z-c_k)$ in a way that differs according to whether c_k is an initial or final point of the kth arc. The solution of the non-homogeneous problem is then expressible in terms of $X(z)$ and an integral over L of $g(t)$.

R. N. Goss (San Diego, Calif.)

1764:

Krzyż, Jan; and Radziszewski, Konstanty. Isoperimetrical defect and conformal mapping. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 10 (1956), 49–56 (1958). (Polish and Russian summaries)

Let $f(z)$ be a function regular for $|z|<R$, and let $L(r)$ denote the length of the image under f of $|z|=r$, and $S(r)$ the area of the image of $|z|<r$, account being taken of multiplicities. Biernacki and Krzyż had previously shown [same Ann. 9 (1955), 135–147; MR 19, 736] that $L^2(r)-4\pi S(r)$ is either a strictly increasing function of r or else vanishes identically. Biernacki and Krzyż further conjectured that $L^2(r)/4\pi S(r)$ is an increasing function of r. The present paper constructs a counter-example to this conjecture. The function in the counter-example does not map the unit circle onto a convex domain, and so the question of the monotonicity of $L^2(r)/S(r)$ for mappings onto convex domains is still open.

H. L. Royden (Zürich)

1765:

Lehman, R. Sherman. Development of the mapping function at an analytic corner. Pacific J. Math. 7 (1957), 1437–1449.

Let D be a domain on the w-plane, the boundary of which in a neighborhood of the origin consists of portions of two analytic curves forming a corner at the origin with opening $\pi\alpha>0$. Let $w=F(z)$ with $F(0)=0$ be a function which maps conformally the upper half-plane onto D. The author establishes an asymptotic expansion of $F(z)$ valid for $z\to 0$ in any finite sector. If α is irrational, then $F(z)\sim\sum A_{kl}z^{k+l\alpha}$ ($k\geqq 0$, $l\geqq 1$; $A_{01}\neq 0$) and if $\alpha=p/q$, a fraction reduced to lowest terms, then

$$F(z)\sim\sum A_{klm}z^{k+l\alpha}(\log z)^m$$

($k\geqq 0$, $1\leqq l\leqq q$, $0\leqq m\leqq k/p$; $A_{010}\neq 0$); the terms in the series are supposed to be arranged in an appropriate order.

Y. Komatu (Tokyo)

1766:

Ou, Šo-Mo. Über den Habitus der konformen Abbildung am Rande des Abbildungsbereiches. Acta Math. Sinica 7 (1957), 271–276. (Chinese. German summary)

This is a Chinese version with German summary of the paper reviewed below.

1767:

Ou, Šo-Mo. On some boundary properties of conformal mapping. Sci. Sinica 7 (1958), 131–136.

Let $f(x)$ be a real, continuous function on the interval $[a, b]$. Denoting by $\omega(\delta; f)$ the modulus of continuity of f, if

$$\int_0^\epsilon \frac{\omega(\delta;f)}{\delta}\,d\delta<\infty \text{ or } \int_0^\epsilon \frac{\omega(\delta;f)}{\delta}\log\frac{1}{\delta}\,d\delta<\infty,$$

$f(x)$ is said to belong to the classes Λ_x, Λ_x^*, respectively. If $f(x)\in L^p$, $p\geqq 1$, and

$$\omega_p(\delta;f)=\sup_{|h|\leqq\delta}\left(\int_a^b |f(x+h)-f(x)|^p dx\right)^{1/p},$$

where $f(x)$ is extended periodically over the whole real line, then the classes of functions defined by the analogous conditions on $\omega_p(\delta; f)$ are denoted by $\Lambda_x(p)$, $\Lambda_x^*(p)$, respectively. Let the simply connected region B in the ζ-plane be bounded by a closed, rectifiable Jordan curve which possesses a tangent at every point. The angle Θ which the tangent makes with the positive axis of reals may be regarded as a function of the arc length s of the curve: $\Theta=\Theta(s)$. Let $\zeta=\varphi(z)$ be a function analytic and schlicht in $|z|<1$ which maps $|z|<1$ conformally onto B. The author proves two theorems. Th. 1: Let n be any natural number or zero. If $\Theta^{(n)}(s)\in\Lambda_s$, then $\varphi^{(n+1)}(z)$ is continuous in $|z|\leqq 1$ and $\varphi^{(n+1)}(e^{i\theta})$ satisfies the inequality $\omega(\delta;\varphi^{(n+1)})\leqq$

$$A\int_0^\delta \frac{\omega(x;\Theta^{(n)})}{x}\,dx+B\delta\int_\delta \frac{\omega(x;\Theta^{(n)})}{x^2}\,dx+C\delta\log\frac{1}{\delta},$$

where A, B, C are constants. Moreover, $\varphi'(z)\neq 0$ in $|z|\leqq 1$.

Th. 2: If $\Theta^{(n-1)}(s)$, $n\geqq 1$, is absolutely continuous, $\Theta^{(n)}(s)\in\Lambda_\sigma(p)$, $p>1$, then $\varphi^{(n+1)}(e^{i\theta})$ satisfies the inequality

$$\omega_p(\delta;\varphi^{(n+1)})\leqq$$

$$A\int_0^\delta \frac{\omega_p(x;\Theta^{(n)})}{x}\,dx+B\delta\int_\delta \frac{\omega_p(x;\Theta^{(n)})}{x^2}\,dx+C\delta^{1/q},$$

where A, B, C are constants and $1/p+1/q=1$. Th. 1 represents an extension of theorems of Kellogg [Trans. Amer. Math. Soc. **13** (1912), 109–132], Warschawski [Nachr. Ges. Wiss. Göttingen **25** (1932), 73–86], Geronimus [Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 889–891; MR **16**, 686], and Alper [Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 423–444; MR **17**, 729]. Th. 2 extends results of Seidel [Math. Ann. **104** (1931), 182–243] and Smirnoff [Math. Ann. **107** (1932), 313–323].

W. Seidel (Notre Dame, Ind.)

1768:

Hažaliya, G. Ya. On an approximate formula in the theory of conformal mappings. Mat. Sb. N.S. **45(87)** (1958), 31–50. (Russian)

The present paper is concerned with approximation formulas which pertain to normalized conformal maps of strip domains whose boundaries are subject to certain conditions of smoothness and other requirements of a special nature. The formulas in question are for the modulus of the limit of the derivative of the mapping function at points of the boundary, and are concerned in particular with estimates near a point of a boundary curve where the curvature is discontinuous. The detailed statement of the assumptions made and of the formulas in question would be lengthy. The results of the author constitute an extension of related results of Lavrentieff [Conformal mappings with applications to some questions of mechanics, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1946]. *M. H. Heins* (Urbana, Ill.)

1769:

Hamburger, L.; Dincă, Fl.; and Manea, V. Sur la torsion de certaines barres cylindriques. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **8** (1957), 1091–1100. (Romanian. Russian and French summaries)

Dans la première partie du travail, on construit la fonction qui réalise la transformation conforme de l'intérieur du cercle unitaire sur l'intérieur d'un profil, ayant pour frontière deux courbes lisses, qui s'entrecoupent sous deux angles différents.

Dans la seconde partie, on indique une méthode de calcul approximative, utile dans le cas où l'intégrale de Schwartz, donnant la solution du problème, ne peut être effectuée sous forme fermée. *Résumé de l'auteur*

1770:

Konstantinesku, Korneliu. [Constantinescu, C.] On the behavior of analytic functions at boundary elements on Riemann surfaces. Rev. Math. Pures Appl. **2** (1957), 269–276. (Russian)

Let R denote an end of a Riemann surface which is of type O_{HB} (resp. O_{HD}) and let p denote an ideal boundary element of R in the sense of Kérékjarto-Stoilow which is of zero capacity in the sense of Sario. It is shown that a single-valued analytic function on R which is bounded (resp. has a bounded Dirichlet integral) possesses a limit at p. An anterior result of this kind was given by the reviewer for the case where the end R has precisely one ideal boundary element [Ann. of Math. (2) **55** (1952), 296–317; MR **13**, 643; p. 302]. *M. H. Heins* (Urbana, Ill.)

1771:

Heins, Maurice. On certain meromorphic functions of bounded valence. Rev. Math. Pures Appl. **2** (1957), 263–267.

If E is a closed set in the plane with the property that there are no non-constant bounded analytic functions in the complement of E, then for each open set $O\supset E$ every bounded analytic function in $O-E$ can be extended to be an analytic function in all of O. Thus the AB-null sets in the plane are removable so far as the bounded analytic functions are concerned.

If one has a Riemann surface of infinite genus, it is natural to ask in what sense is its boundary "removable"? Heins [Ann. of Math. (2) **55** (1952), 296–317; MR **13**, 643] established the following theorem, which seems to the reviewer to be the prototype of the sort of thing one should expect of "removability": Let F be an open Riemann surface with a compact relative boundary Γ and an ideal boundary consisting of just one end and that of harmonic measure zero. Then there is a mapping φ of F into a finite Riemann surface G such that every bounded analytic function f on F is the composition $g\circ\varphi$ of φ with a bounded analytic function on G. (It follows from recent results of Wermer's on function algebras that the hypothesis of this theorem can be weakened to approximately the requirement that each bounded analytic function on F assumes its maximum on Γ.)

In the present paper Heins gives another criterion for a boundary to be removable in the above sense: Given a Riemann surface F, and suppose that there exists a non-constant meromorphic function f_0 of bounded (finite) valence such that $\{w|\nu_{f_0}(w)=\max \nu_{f_0}\}\in O_{AB}$, where $\nu_{f_0}(w)$ denotes the number of times f_0 assumes the value w on F. Then there is a conformal mapping φ of F into a compact Riemann surface G, such that every function f on F of bounded valence is the composition $g\circ\varphi$ of φ with a rational function g on G. Further, for each function f of bounded valence on F we have $\{w|\nu_f(w)=\max \nu_f\}\in O_{AB}$.

The methods of Heins rely heavily on the theory of interior transformations and on the theory of algebraic structure. They will probably play a fruitful rôle in the further study of the "removability" of the ideal boundary of Riemann surfaces. *H. L. Royden* (Zürich)

1772:

Teleman, C. Une classe de fonctions analytiques d'une surface de Riemann, généralisant les intégrales abéliennes. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 163–182. (Romanian. Russian and French summaries)

The author considers multiple-valued analytic functions h on a Riemann surface whose determinations are related by linear fractional transformations and also are locally inverses of local uniformizers. It is shown that if h_1 and h_2 are such functions, $\{h_1\}_{h_2}dh_2^2$ is a regular quadratic differential on R, $\{h_1\}_{h_2}$ being the Schwarzian derivative of h_1 with respect to h_2; it is skew symmetric in (h_1, h_2). The h are determined for special algebraic curves. Geometric interpretations of the functions h are given. Applications to the problem of space forms with constant negative curvature are to be given in a subsequent paper.

M. H. Heins (Urbana, Ill.)

1773:

Tietz, Horst. Funktionen mit Cauchyscher Integraldarstellung auf nicht-kompakten Gebieten Riemannscher Flächen. Ann. Acad. Sci. Fenn. Ser. A. I, no. **250/36** (1958), 10 pp.

This paper constitutes a continuation of the author's paper [Math. Ann. **132** (1957), 412–429; MR **18**, 883]. The basic notions involved are Laurent separations and elementary differentials taken in the sense of Behnke and Stein [Math. Ann. **120** (1948), 430–461; MR **10**, 696]. It is shown that a Laurent separation determines a unique elementary differential dF which induces in a specified way the given Laurent separation. It is further shown that the functions and differentials of certain classes associated with a Laurent separation are precisely those which admit a Cauchy integral representation in terms of dF. Under certain circumstances "Fabergroessen" (which are analytic functions or differentials) may be introduced to yield expansions for members of classes of functions associated with the given Laurent separation. This is applied, in particular, to elementary differentials.

M. H. Heins (Urbana, Ill.)

1774:

Huckemann, Friedrich. Über den Defekt von mittelbaren Randstellen auf beschränktartigen Riemannschen Flächen. Ann. Acad. Sci. Fenn. Ser. A. I, no. **250/16** (1958), 12 pp.

Let $w(z)$ be meromorphic and of bounded characteristic in $|z|<1$. Let Γ_w be a closed set containing for almost all φ the radial limits $w(e^{i\varphi})$ of $w(z)$, and let $\mathfrak{A}$ denote the class of functions $w(z)$ for which Γ_w is not necessarily the whole plane. The reviewer has shown [Ann. Acad. Sci. Fenn. Ser. A I, no. **177** (1954); MR **16**, 688] that for the class $\mathfrak{A}$, a value distribution theory can be developed which exhibits analogous features with the Nevanlinna theory for functions of unbounded characteristic. E.g., a concept of deficiency $\delta(a)$ can be introduced; in the particular case that the set of values of $w(z)$ is a domain G disjoint with Γ_w, we have $\delta(a)=1-\lim_{r\to 1} N(r, a)/g(a, w(0), G)$, where N is the counting function and g the Green's function of G. The boundary points of the Riemann surface generated by $w(z)$ can be classified, after the classical example of parabolic surfaces, into direct and indirect critical points. The reviewer proved [loc. cit.] that a direct critical point always corresponds to a value with positive deficiency and posed the question whether an indirect critical point, too, can give rise to a positive deficiency. This question is answered in the affirmative in the present paper. The author's example is the indirect critical point over $w=\infty$ of the Riemann surface obtained from the surface generated by $w=(\cos\sqrt{z})^{-1}$ by shifting the branch points from $w=\pm 1$ to $w=(-1)^{\nu}a_{\nu}$, $a_{\nu}>1$, where $a_{\nu}\to\infty$ with sufficient rapidity. The result shows a certain analogy with the situation in the theory of parabolic surfaces, where indirect critical points of similar type give rise to values with positive deficiency.

D. Lehto (Helsinki)

1775:

Schlesinger, Ernest C. Conformal invariants and prime ends. Amer. J. Math. **80** (1958), 83–102.

The author gives a new definition of prime ends of a simply connected domain Ω. For crosscuts, he uses continuous open arcs whose closures need not be continuous (this departure from convention is made necessary by the conformal invariance of the treatment). With a sequence $\{a_n\}$ of points in Ω, he associates the family (γ) of crosscuts that separate almost all points a_n from a closed subset of Ω. A sequence $\{a_n\}$ is a P-sequence provided the associated family (γ) has extremal length zero; two P-sequences are equivalent provided their union is a P-sequence; and finally, the prime ends of Ω (in the classical sense) can be regarded as the equivalence classes of P-sequences.

G. Piranian (Ann Arbor, Mich.)

1776:

Ohtsuka, Makoto. On boundary cluster sets of functions analytic in the unit circle. Rev. Math. Pures Appl. **2** (1957), 317–321.

Let $f(z)$ be a meromorphic function in $U=\{z:|z|<1\}$. The cluster set and the range of values of $f(z)$ at a point $z_0\in C=\{z:|z|=1\}$ are denoted by S_{z_0}, R_{z_0}, respectively. Ohtsuka [Trans. Amer. Math. Soc. **78** (1955), 294–304; MR **16**, 686] and Noshiro [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 398–401; MR **17**, 143] have defined certain boundary cluster sets of $f(z)$ at z_0. The author shows, by means of a theorem of Bagemihl [ibid. **41** (1955), 379–382; MR **16**, 1095], that the smallest of these boundary cluster sets is one defined by Ohtsuka: $S\Gamma_{z_0}$. From this it follows that two theorems of Ohtsuka and two of Noshiro, concerning the existence of asymptotic values of $f(z)$ at z_0 or at points $z_n\in C$ tending to z_0, are included in two other theorems of Ohtsuka of the same kind (each of these six theorems appearing in one of the first two papers cited above), which, in the special case that S_{z_0} is the whole plane, are essentially the same as the following theorem proved in the present paper. Let $f(z)$ be meromorphic in U; suppose that S_{z_0} is the whole plane and that $w_0\notin R_{z_0}$. If (C_1) there exists, for some $\rho>0$, a path L outside $N_{\rho}(C^2)$ (a certain set used in defining $S\Gamma_{z_0}$) converging to w_0, and (C_2) there exists a set $E\subset C$, dense in some neighborhood of z_0, such that, for every $z\in E$, there is a path $\Lambda(z)$ in U terminating in z, with the property that the cluster set of $f(z)$ at z along $\Lambda(z)$ does not contain w_0, then w_0 is an asymptotic value of $f(z)$ at z_0 or at points $z_n\in C$ tending to z_0. Conditions (C_1) and (C_2) are necessary in this theorem. The author concludes with some remarks about the case where $f(z)$ assumes values on an abstract Riemann surface.

F. Bagemihl (Notre Dame, Ind.)

1777:

Biernacki, Mieczysław. Sur les moyennes et les extréma des modules des fonctions analytiques. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **10** (1956), 127–136 (1958). (Polish and Russian summaries)

This paper consists of two parts. In the first part the author proves that if $f(z)$ is regular in $|z|<R$, and for $|z_1|<R$, $|z_2|<R$ we always have

$$|f(z_2)|<\lambda^{2\lambda/(\lambda-1)}|f(z_1)|,$$

where $\lambda>1$, then

$$\int_0^{2\pi}|f(re^{i\theta})|^{\lambda}d\theta\Big/\int_0^{2\pi}|f(re^{i\theta})|d\theta$$

increases with r. He applies this to schlicht functions and their derivatives.

In the second part the author proves that if $M(r)$ denotes the maximum modulus of $f(z)$ and z_0 is a point where $M(r)$ is attained, then

$$|f''(z_0)|\geq\frac{M'(r)}{r}\left(r\frac{M'(r)}{M(r)}-1\right).$$

Here the right hand side is positive, except in certain cases indicated by the author.

W. K. Hayman (London)

1778:

Lewandowski, Zdzisław. Nouvelles remarques sur les théorèmes de Schild relatifs à une classe de fonctions univalentes (Démonstration d'une hypothèse de Schild). Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **10** (1956), 81–94 (1958). (Polish and Russian summaries)

Let $p(z)=z-\sum_2^N a_n z^n$, $N\geqq 2$, be a polynomial with non-negative coefficients a_n, and such that $|z|<1$ is the largest open disc, centre 0, in which $p(z)$ is univalent. Let $|w|<d^*$ be the largest disc contained in the map of $|z|<1$, and $|w|\leqq d^0$ be the largest disc contained in the map of $|z|\leqq r_0$, where $|z|\leqq r_0$ is the largest disc of convex mapping. The author proves that $d^0\geqq\frac{1}{2}d^*$ with equality for $p(z)=z-\frac{1}{2}z^2$. He thus verifies a conjecture of A. Schild [Proc. Amer. Math. Soc. **5** (1954), 115–120; MR **15**, 694] who proved that $d^0>\frac{1}{3}d^*$.

W. W. Rogosinski (Newcastle-upon-Tyne)

1779:

Hummel, J. A. A variational method for starlike functions. Proc. Amer. Math. Soc. **9** (1958), 82–87.

Let S denote the class of analytic functions $f(z)$ which are univalent and starlike with respect to the origin in $|z|<1$, and are normalized by $f(0)=0$, $f'(0)=1$. By modifying Schiffers' variational method in a suitable way, the author derives a formula for a variation of $f(z)$ which does not lead out of the class S. This formula is then used to obtain a geometric characterization of the functions solving a certain type of extremal problem within S. As the author remarks, this result may also be proved by an appeal to the Pick-Nevanlinna interpolation problem.

Z. Nehari (Pittsburgh, Pa.)

1780:

Hummel, J. A. The coefficient regions of starlike functions. Pacific J. Math. **7** (1957), 1381–1389.

Let S^* be the class of functions $f(z)=z+a_2z^2+a_3z^3+\cdots$, regular, univalent and starlike in the unit circle. (A necessary and sufficient condition for the last mentioned property is that $\Re\{zf'(z)/f(z)\}$ be positive throughout $|z|<1$.) In an earlier paper [#1779 above], the author described a variational method for functions of class S^*. In the present paper, he uses this method to obtain, in explicit terms, the coefficient regions of functions of S^*. Let V_n^* be the $(2n-2)$ dimensional region composed of those points $(a_2, a_3, \cdots, a_n)$ which belong to the functions of S^*. The equation of the boundary of V_n^* is then given in terms of certain quantities $A_{j,k}$ and $B_{j,k}$, which are polynomials in the a_ν and their conjugates and which are easily obtainable by simple recursion formulas. In particular, if $(a_2, a_3, \cdots, a_{n-1})$ is an interior point of V_{n-1}^* then the cross section of V_n^*, with $a_2, a_3, \cdots, a_{n-1}$ held fixed, is a circular disc whose equation is $|A_{1,n-1}|\leqq B_{1,n-1}$.

F. Herzog (East Lansing, Mich.)

1781:

Kulshrestha, P. K. On evaluations of the measure of curvature of level curves of schlicht functions. Ganita **7** (1956), 123–137.

Let $f(z)$ be regular and univalent in $|z|<1$, and normalized by $f(0)=0$, $f'(0)=1$. The author determines the exact lower and upper bounds for the curvature of the image of the circumference $|z|=r$ $(0<r<1)$ under the mapping $z\to f(z)$. The proof is based on the approximation of the image region by polygons and on the explicit representation of the mapping functions available in this case.

Z. Nehari (Pittsburgh, Pa.)

1782:

Gel'fer, S. A. On the maximum of the conformal radius of the fundamental region of a given group. Mat. Sb. N.S. **44(86)** (1958), 213–224. (Russian)

In 1934, Lavrent'ev [Trudy Fiz.-Mat. Inst. Steklov. **5**, 159–245] considered the properties of the functions $f(z)$ which maximize $|f'(0)|$ in the class of functions S_a: $w=f(z)=\sum_1 c_n z^n$, regular and univalent for $|z|<1$, and omitting the preassigned points a: $a_1, a_2, \cdots, a_m$. The author considers the same problem for the class $S_a(\omega_1,\omega_2)\subset S_a$ of $f(z)$ which map $|z|<1$ onto some subregion of some fundamental region of the group of transformations T_n: $w\to w+n_1\omega_1+n_2\omega_2$, n_i integral. This problem is non-trivial even for the class $S(\omega_1,\omega_2)$ corresponding to the case when a is empty. By using a Golusin-type variation he finds that the extremal function satisfies $z^{-2}w'^{-2}=A_0+\sum_1^m A_i\zeta(w-a_i)+A_{m+1}\zeta(w)+\rho(w)$, where ρ, ζ are the elliptic and zeta functions of Weierstrass, and the A_i are certain constants. He proves the following. Th. 1: The extremal function maps $|z|<1$ onto a fundamental region S_0 of $\{T_n\}$, bounded by a finite number of analytic arcs, minus a finite number of piecewise analytic slits emanating from $a_1, \cdots, a_m$, in such a way that parts of bdry(S_0), congruent under $\{T_n\}$, and opposite sides of slits correspond to arcs of equal length on $|z|=1$. Th. 2: The function satisfying the statement of Th. 1 is essentially unique.

The cases when the invariants g_2, g_3 of $\rho(w)$ are real are considered in more detail. In particular, the special cases (1) $\omega_1>0$, $\omega_2=\infty$, $m=2$, a_i imaginary, (2) $\omega_1>0$, $\omega_2=\infty$, $m=1$, $a_1>0$, and (3) a empty, are discussed. The region S_0 for which the extremal is attained is not always a polygon.

E. Reich (Minneapolis, Minn.)

1783:

Lewis, C. J. The problem of Milloux for functions analytic in an open annulus. Duke Math. J. **25** (1958), 591–600.

Let A be the annulus defined by $s<|z|<1$, where $0<s<1$. Let E be a point set in A, closed with respect to A, with the property that it intersects all circles $|z|=\rho$ lying in A. Let $0<m<1$ and $s<r<1$, and let $\mathfrak{F}_m$ be the family of functions g satisfying 1) g is analytic in A, 2) $|g|<1$ in A, 3) there is some set E on which $|g|\leqq m$. The problem of Milloux is to determine sup max $|g(z)|$, where the maximum is taken over all z such that $|z|=r$, and the supremum over all $g\in\mathfrak{F}_m$.

That extremal functions exist follows from considerations of normal families; let these extremal functions f be so normalized that $f(r)=\max|f(z)|$ for $|z|=r$. Then the following results are obtained: f is unique, is independent of r, is real for real z, and has the segment $-1<|z|<-s$ as its E-set. Specific representations of f in terms of θ functions and other special functions are given. The problem is related to the corresponding problem for the circle by showing that $f=\phi(\psi(z))$, where ψ is a specified map of the annulus on the circle, and ϕ the solution of the Milloux problem for the circle.

J. W. Green (Los Angeles, Calif.)

1784:

McKiernan, Michel A. Séries d'itérateurs et leurs applications aux équations fonctionnelles. C. R. Acad. Sci. Paris **246** (1958), 2331–2334.

The author studies the iteration of functions from the point of view of K. Menger [Calculus, a modern approach, Ginn, New York, 1955; MR **17**, 351] and applies the

resulting formalism to solve problems like finding f from $f(g)-f=h$, knowing g and h.

H. P. McKean, Jr. (Cambridge, Mass.)

1785:

McKiernan, Michel A. Le prolongement analytique des séries d'itérateurs. C. R. Acad. Sci. Paris **246** (1958), 2564–2567.

The author continues his study of iterates of functions of a complex variable [see review above]. Given a power series $g(z)=\sum a_n z^n$ converging near $z=0$, consider the "analytic operator" $\mathbf{A}: f \to \sum a_n f^{[n]} = (2\pi i)^{-1} \int \varphi(\cdot, z) g(z) dz$, where $f^{[0]}(z)=z$, $f^{[1]}(z)=f(z)$, $f^{[2]}(z)=f(f(z))$, etc., $\varphi(\cdot, z)=\sum z^{-n-1} f^{[n]}$, and the integral is taken over a small circle with center at $z=0$. The paper is devoted to the "analytic continuation" of such operators and to the "product" $\mathbf{A}_1 \hat{\times} \mathbf{A}_2$: $f \to (2\pi i)^{-1} \int \varphi(\cdot, z) g_1(z) g_2(z) dz$ of two such. The "product" was given a different definition in the note cited. *H. P. McKean, Jr.* (Cambridge, Mass.)

1786:

Cowling, V. F. Series of Legendre and Laguerre polynomials. Duke Math. J. **25** (1957), 171–176.

Let $P_n(z)$ denote the Legendre polynomial of order n, and let the series $f(z)=\sum_{n=0}^{\infty} a_n P_n(z)$ converge in a finite ellipse in the complex z-plane. The problem treated in this paper is that of determining regions outside the original ellipse of convergence to which $f(z)$ can be continued analytically. The author succeeds in finding such regions in the case in which $a_n=g(n)$, where $g(z)$ is an analytic function with a certain prescribed behavior. Similar results are obtained for series of Laguerre polynomials.

Z. Nehari (Pittsburg, Pa.)

1787:

Arsove, Maynard G. The Pincherle basis problem and a theorem of Boas. Math. Scand. **5** (1957), 271–275.

If $\{q_n\}$ is a basis for a linear space, and p_n is not too far away from q_n, then $\{p_n\}$ too will be a basis. Explicit theorems of this nature were proved by Paley and Wiener for Hilbert space, and by Boas for a Banach space [R. P. Boas, Jr., Trans. Amer. Math. Soc. **48** (1940), 467–487; MR 2, 80] Several years ago, the reviewer observed that these were instances of the simple remark that if $\sum T^n$ converges, $(I-T)^{-1}$ exists. [Lectures on functions of a complex variable, Univ. of Mich. Press, Ann Arbor, 1955, pp 409–419; MR **17**, 140]. The author rediscovers this, and applies it to show that $\alpha_n(z)=z^n(1+\lambda_n(z))$ is a basis in the space of functions analytic on the disc $|z|<R$ if

$$\lim_{r \to R} \limsup_{n} \max_{|z| \leq r} |\lambda_n(z)| < 1.$$

R. C. Buck (Stanford, Calif.)

1788:

Padmanabhan, K. S. On generalized Taylor expansions. Math. Student **25** (1957), 153–155.

The author discusses the expansion properties of functions $F_n(z)=z^n(1+\lambda_n(z))$ by explicit substitution and rearrangement of power series. {His results are not as good as those that may be obtained more easily by more general methods; see the paper reviewed above or Boas, Trans. Amer. Math. Soc. **48** (1940), 467–487 [MR 2, 80].}

R. C. Buck (Stanford, Calif.)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 1759, 1911.

1789:

Schieferdecker, Eberhard. Ein Analogon des Satzes von Pick in der Theorie mehrerer komplexer Veränderlichen. Arch. Math. **8** (1957), 115–124.

The author first proves an analogue of the Schwarz Lemma for bounded holomorphic mappings of the unit hypersphere K^n in C^n into the unit hypersphere K^m in C^m. He then uses a metric which is invariant with respect to the holomorphic automorphisms of the unit hypersphere to prove an analogue of the Pick theorem which says that in a holomorphic mapping of the unit hypersphere in C^n into the unit hypersphere in C^m, the noneuclidean distance between any two points is not increased. If there is a single point X in K^n such that on any noneuclidean complex line through X, there is some other point Y in K^n whose noneuclidean distance from X remains the same under the mapping, then all noneuclidean distances remain unchanged and the mapping is a holomorphic automorphism of K^n.

G. Springer (Lawrence, Kans.)

1790:

Sakae, Eiichi. Domains of regularity. Sûgaku **9** (1957/58), 17–44. (Japanese)

Expository article on the recent progress made by various authors on domains of regularity (in the theory of several complex variables). Chap. 1, Convexity of domains; Chap. 2, Theorems on continuation and approximation; Chap. 3, The problem of Levi.

Rimhak Ree (Vancouver, B.C.)

1791:

Hitotumatu, Sin. On the problem of Cousin. Sûgaku **8** (1956/57), 102–117. (Japanese)

Expository article showing how the classical problem of Cousin (in the theory of several complex variables) has been solved by using sheaves.

Rimhak Ree (Vancouver, B.C.)

1792:

Miyazaki, Kenichi. On a theorem of Ehrenpreis. J. Sci. Hiroshima Univ. Ser. A **21** (1957/58), 115–117.

Let $f(z_1, \cdots, z_n)$, $g(z_1, \cdots, z_n)$ be functions which are analytic in the neighborhood of $(0, \cdots, 0)$. We say that the cospectrum of g contains that of f at $(0, \cdots, 0)$ if there exists a neighborhood N of $(0, \cdots, 0)$ so that both f and g are analytic in N and for any $P \in N$ and any nonnegative integers $t_1, \cdots, t_n$ the conditions

$$[(\partial^{s_1+\cdots+s_n}/\partial z_1^{s_1} \cdots \partial z_n^{s_n}) f](P)=0$$

whenever $s_1 \leq t_1, \cdots, s_n \leq t_n$ imply the same for g.

The reviewer has proved the following result: If f is not identically zero and the cospectrum of g contains that of f at $(0, \cdots, 0)$ then g/f is analytic in a neighborhood of $(0, \cdots, 0)$. [Amer. J. Math. **77** (1955), 293–328; MR **16**, 1122].

The author gives a new proof of this theorem by using a suitable linear transformation combined with the classical Weierstrass preparation theorem, whereas the reviewer's original proof depended on a modified form of the Weierstrass preparation theorem. The difficulty in the proof comes from the fact that it is not a priori clear that the condition: "cospectrum of g contains that of f at $(0, \cdots, 0)$" is invariant under linear transformations.

L. Ehrenpreis (Waltham, Mass.)

1793:

Gundlach, Karl-Bernhard. Über den Rang der Schar der ganzen automorphen Formen zu hyperabelschen Transformationsgruppen in zwei Variablen. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. **1958**, 59–66.

The following theorem was proved by Hervé [Ann. Sci. Ecole Norm. Sup. (3) **69** (1952), 277–302; MR **14**, 633]: Let Γ be a discontinuous group of transformations on a bounded region of the space of two complex variables, and let Γ have a compact fundamental region. Then the dimension of the vector space of automorphic forms of dimension $-r$ ($r>0$) is a quadratic polynomial in r, provided r is sufficiently large. In the present paper the author extends this result to a class of hyperabelian transformation groups in two complex variables whose fundamental regions are not compact.

J. Lehner (East Lansing, Mich.)

1794:

Braun, Hel. Darstellung hermitischer Modulformen durch Poincarésche Reihen. Abh. Math. Sem. Univ. Hamburg **22** (1958), 9–37.

In the theory of automorphic forms in several complex variables, the so-called representation theorems have been achieved only for the Siegel modular group [Maass, Math. Ann. **123** (1951), 125–151; MR **13**, 210]. In previous papers the author has discussed the Hermitian modular group of degree n and its automorphic forms [Ann. of Math. (2) **50** (1949), 827–855; **51** (1950), 92–104; **53** (1951), 143–160; Abh. Math. Sem. Univ. Hamburg **19** (1955), 134–148; MR **11**, 333; **12**, 482; **16**, 801], and Becker has introduced Poincaré series on this group and its congruence subgroups [Math. Ann. **129** (1955), 187–208; MR **17**, 603]. In the present paper, she establishes a representation theorem for the congruence subgroups of the Hermitian modular group of degree n, and so, in particular, for the full group. She proves, in fact, that every automorphic form of sufficiently high negative dimension k ($-k \geqq 4n$, k=even integer) is a finite linear combination of Poincaré series.

The principal tool used in the proof is the scalar product of two automorphic forms as developed by Maass [cf. above] and Koecher [Math. Z. **59** (1954), 399–416; **61** (1955), 455–466; MR **15**, 603; **16**, 801]. The representation theorem is first proved for cusp forms. The extension to the general case is then made by an inductive argument using the Siegel operator Φ, which maps forms of degree n into those of degree $n-1$.

J. Lehner (East Lansing, Mich.)

1795:

Yano, Kentaro. On analytic vectors. Sûgaku **8** (1956/57), 193–206. (Japanese)

It is well known that in a compact Kähler space a covariant analytic vector is a harmonic vector and conversely [G. de Rham and K. Kodaira, Harmonic integrals, Inst. Advanced Study, Princeton, 1950; K. Yano and S. Bochner, Curvature and Betti numbers, Annals of Math. Studies, no. 32, Princeton, 1950; MR **12**, 279; **15**, 989]. However, we know not so many facts concerning contravariant analytic vectors. The purpose of the paper is to explain some results concerning contravariant analytic vectors which were obtained by the author [Michigan Math. J., under press; also The theory of Lie derivatives and its applications, North-Holland Publ. Co., Amsterdam 1957; MR **19**, 576], and to show some new results related to them.

A. Kawaguchi (Sapporo)

1796a:

Obata, Morio. Affine connections on manifolds with almost complex, quaternion or Hermitian structure. Jap. J. Math. **26** (1956), 43–77.

1796b:

Obata, Morio. Affine connections in a quaternion manifold and transformations preserving the structure. J. Math. Soc. Japan **9** (1957), 406–416.

1796c:

Obata, Morio. Hermitian manifolds with quaternion structure. Tôhoku Math. J. (2) **10** (1958), 11–18.

The three papers under review are an exhaustive study of connections in almost complex, quaternion and Hermitian structures, a large portion of which is an elaboration of the theorem by A. Frölicher [Math. Ann. **129** (1955), 50–95; MR **16**, 857] stating that there always is an affine connection with respect to which an almost complex structure is covariant constant. To a large extent, similar, but more refined methods are used, and intensive use is made of projection operators introduced by Schouten and Yano [Indag. Math. **17** (1955), 1–9; MR **16**, 858]. Since there are a total of 52 theorems in the papers (not all original, as indicated by appropriate references to the 37-title bibliography) only a few points can be illustrated here.

A quaternion structure on a manifold is a pair of almost-complex structures $\phi_j{}^i$, $\psi_j{}^i$, satisfying the additional condition $\phi_h{}^i\psi_j{}^h+\psi_h{}^i\phi_j{}^h=0$. The dimension n on the manifold has to be $4m$, and $\kappa_j{}^i=\phi_h{}^i\psi_j{}^h$ is another almost-complex structure. The three tensors ϕ, ψ, κ give the action of the units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ of the real quaternions on the tangent space. There are three torsion tensors $N_{ij}{}^h(\phi)$, $N_{ij}{}^h(\psi)$, $N_{ij}{}^h(\kappa)$ which are related by $N_{ij}{}^h(\phi)=2\Phi_2\Phi_3(N_{ij}{}^h(\psi)+N_{ij}{}^h(\kappa))$, cycl.; where the Φ's are projection operators corresponding to the almost complex structure ϕ. The quaternion structure is integrable if and only if $N(\phi)$, $N(\psi)$ vanish; and, in addition, a certain tensor M is zero. M is given by $M_{\bar{\gamma}\mu\lambda}{}^{\kappa}=\partial_{\bar{\gamma}}((\partial_\mu\psi_\lambda{}^{\bar{\alpha}})\psi_{\bar{\alpha}}{}^{\kappa})$, conj., an expression which becomes meaningful only after complex coordinates compatible with ϕ have already been introduced. — Another result: the group of transformations preserving the quaternion structure is a Lie group (being a closed subgroup of a group of affine transformations).

For almost-complex structures an exhaustive study is made of the torsion of connections in which ϕ is covariant constant (ϕ-connections). Necessary and sufficient conditions are established that a given tensor field $S_{ij}{}^h$ can be the torsion tensor of some ϕ-connection. The most interesting special case is $N_{ij}{}^h=2S_{ij}{}^h$.

Next, the same problem is considered for (ϕ, ψ)-connections in quaternion structures. The pattern of results is largely the same. Special study is made of complex analytic manifolds with quaternion structure, i.e., of structures in which ϕ is integrable. It turns out that the tensor M satisfies $M=-R$, where R is the curvature of some specially suitable (ϕ, ψ)-connection.

Finally, the pattern is repeated for almost-Hermitian and almost-Hermitian quaternion structures. Among the results: If a Kählerian manifold has a quaternion structure, then the Ricci curvature vanishes.

{The author was not familiar with Newlander and Nirenberg's recent result [Ann. of Math. (2) **65** (1957), 391–404; MR **19**, 577], whence he frequently has to make the assumption of analyticity, or use the prefix 'pseudo' in some places. Also, the author seemed unfamiliar with

Lichnerowicz' method [Théorie globale des connexions et des groupes d'holonomie, Edizioni Cremonese, Rome, 1957; MR 19, 453], which could have reduced the mechanical computations considerably.}

A. Nijenhuis (Seattle, Wash.)

SPECIAL FUNCTIONS

See also 1786, 1876.

1797:

Gurland, John. An inequality satisfied by the Gamma function. Skand. Aktuarietidskr. **39** (1956), 171–172 (1957).

1798a:

Meynieux, Robert. Méthode de Liouville appliquée à l'équation de Weber. C. R. Acad. Sci. Paris **246** (1958), 3208–3210.

1798b:

Meynieux, Robert. Approximations de la fonction Γ appliquées aux solutions de l'équation de Weber. C. R. Acad. Sci. Paris **246** (1958), 3312–3314.

The asymptotic behavior of Weber functions, solutions of $u''+(n-w^2/4)u=0$, is considered when in and $i(w^2-4n)$ have large real parts, n and w complex. In the first note $D_{n-\frac{1}{2}}(w)$ and $D_{-n-\frac{1}{2}}(iw)$ are discussed; in the second the even and odd solutions are treated. The reader should also consult the more comprehensive work of Erdélyi, Kennedy, and McGregor [J. Rational Mech. Anal. **3** (1954), 459–485; MR **16**, 33].

N. D. Kazarinoff (Ann Arbor, Mich.)

1799:

Muradyan, R. M. Asymptotic formulas for generalized Legendre functions and Gegenbauer functions. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 887–890. (Russian)

Asymptotic expressions are obtained for $P_n^m(\cos\varphi)$, $Q_n^m(\cos\varphi)$, and $C_n^m(\cos\varphi)$ for large $|n|$ and fixed m. The method is to substitute the product of two undetermined functions into Legendre's equation, specializing one function to simplify the equation, and observing that the resulting equation satisfied by the other function is Bessel's equation with the exception of a term whose coefficient is $O(1/n^2)$. In particular, it is found that

$$P_n^m(\cos\varphi)\sim(n+\tfrac{1}{2})^m\Big(\frac{\varphi}{\sin\varphi}\Big)^{\frac{1}{2}}J_{-m}((n+\tfrac{1}{2})\varphi) \text{ as } |n|\to\infty.$$

R. G. Langebartel (Urbana, Ill.)

1800:

Al-Salam, W. A. Note on a q-identity. Math. Scand. **5** (1957), 202–204.

A q-identity due to F. H. Jackson [Quart. J. Math. Oxford Ser. **12** (1941), 167–172; MR **3**, 238] is specialized to yield a q-analogue of an identity in binomial coefficients which includes J. E. Fjeldstad's generalization of Dixon's formula [Math. Scand. **2** (1954), 46–48; MR **16**, 3]. Another binomial identity included is

$$\sum_{s=0}^{2m}(-1)^s\binom{2m}{s}^4\binom{3m+s}{s}\binom{5m}{s}^{-1}=(-1)^m\binom{3m}{m}^3\binom{4m}{m}\binom{5m}{2m}^{-1}.$$

J. Riordan (New York, N.Y.)

1801:

Shukla, H. S. Certain transformations of nearly-poised bilateral hypergeometric series of special type. Canad. J. Math. **10** (1958), 195–201.

From a relation between M nearly-poised hypergeometric series of the type ${}_MH_M$ where

$${}_MH_M\left[\begin{matrix}a_1, a_2, \cdots, a_M;\\ b_1, b_2, \cdots, b_M;\end{matrix}\right]=\sum_{n=-\infty}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_M)_n}{(b_1)_n(b_2)_n\cdots(b_M)_n},$$

the author deduces various special transformations of this type. He obtains a relation between a nearly-poised ${}_4F_3$ series of the first kind and three nearly-poised ${}_4H_4$ series, and he deduces a relation between two Saalschutzian ${}_5F_4$ series and three nearly-poised ${}_4H_4$ series, which is itself a generalization of a result due to W. N. Bailey.

Next the author discusses the corresponding relation between a nearly-poised ${}_5F_4$ series of the first kind and four nearly-poised ${}_5H_5$ series, from which he deduces a relation between a nearly-poised ${}_5F_4$ series of the first kind and three nearly-poised ${}_5H_5$ series, and also a relation between two Saalschutzian ${}_5F_4$ series and three nearly-poised ${}_5H_5$ series, both of which generalize results due to Bailey.

In particular, he gives the sum of the nearly-poised series

$${}_3H_3\left[\begin{matrix}2c-b-1, c, d;\\ b, c-1, a;\end{matrix}\right]$$

and the sum of the nearly-poised series

$${}_3F_2\left[\begin{matrix}2c-2, c, b;\\ c-1, a;\end{matrix}1\right].$$

L. J. Slater (Cambridge, England)

1802:

Vilenkin, N. Ya. Continuous analogue of the addition theorem for Jacobi polynomials. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 2(80), 157–161. (Russian)

The function studied is

$$P_{mn}^{l}(\mu)=A(1-\mu)^{\frac{1}{2}(m-n)}(1+\mu)^{-\frac{1}{2}(n+m)}\times(d/d\mu)^{l-n}[(1-\mu)^{l-m}(1+\mu)^{l+m}],$$

where

$$A=\frac{(-1)^{l-m}i^{n-m}}{2^l(l-m)!}\sqrt{\frac{(l-m)!(l+n)!}{(l+m)!(l-n)!}};$$

the definition can be extended to general l, m, n in terms of a hypergeometric function. The author evaluates an integral involving $P_{m,-s}^{l}(\cos\theta')P_{-n,s}^{l}(\cos\theta'')$ over $-\infty<s<\infty$, the result being analogous to a Bessel-function formula of Ramanujan [see G. N. Watson, A treatise on the theory of Bessel functions, 2nd ed. Cambridge, 1944; MR **6**, 64; p. 449, § 13.8(1)]. The proofs of the two results are also analogous.

F. V. Atkinson (Canberra)

1803:

Singh, V. N. Certain generalized hypergeometric identities of the Rogers-Ramanujan type. II. Pacific J. Math. **7** (1957), 1691–1699.

[Cf. première partie, même J., 1011–1014; MR **19**, 410.] Poursuivant ses recherches relatives aux travaux d'Alder sur la généralisation des identités de Rogers-Ramanujan, l'auteur donne de nouvelles identités contenant des séries hypergéométriques.

R. Campbell (Caen)

1804:

Vilenkin, N. Ya. On a generating function for Jacobi polynomials. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 6(78), 137–142. (Russian)

By the use of group representations, this note gives

some relations between Jacobi polynomials and the polynomials introduced by Bargmann [Ann. of Math. (2) **48** (1947), 568–640; MR **9**, 133].

M. M. Day (Urbana, Ill.)

1805:

Frey, Tamás. On the boundedness of orthogonal polynomials. I. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **8** (1958), 67–87. (Hungarian)

Let $\omega_n(x)$, $\pi_n(x)$ be sets of orthogonal polynomials associated with weight-functions $w(x)$, $p(x)$ on $[-1, 1]$. The author finds conditions on $w(x)$, $p(x)$ which ensure that $\omega_n(x)$, $\pi_n(x)$ have a similar behaviour as $n\to\infty$, bounds being established of the form

$$|\omega_n(\xi)|\leqq C\sum_{\nu=0}^{s}|\pi_{n-\nu}(\xi)|,\quad |\pi_n(\xi)|\leqq C'\sum_{\nu=0}^{s}|\omega_{n-\nu}(\xi)|,$$

where s, C, C' are independent of n. In the result of Korous [see G. Szegö, Orthogonal polynomials, Amer. Math. Soc., New York, 1939; MR **1**, 14; p. 157] we have $s=1$, and $w(x)/p(x)$, $p(x)/w(x)$ are bounded and satisfy a Lipschitz condition. In the present extensions there may be a finite number of points at which these ratios do not satisfy the Lipschitz condition, and possibly become infinite. At such a point x_j there are required bounds of the form

$$p|x-x_j|^{A_j}\leqq p(x)\leqq P|x-x_j|^{a_j},$$
$$w|x-x_j|^{B_j}\leqq w(x)\leqq W|x-x_j|^{b_j},$$

where w, W, p, P are positive constants and $-1<a_j\leqq A_j<a_j+1$, modified to $-1<a_j\leqq A_j<a_j+\frac{1}{2}$ if $x_j=\pm 1$, and similarly for the b_j, B_j. The case of polynomials on the unit circle is also considered, and plays a part in the proof. Finally, the author proves results weakening the Lipschitz condition. For example, if $k(x)=w(x)/p(x)$, we might have

$$|k(x)-k(x_0)|\leqq|x-x_0|^{\frac{1}{2}}\cdot g(|x-x_0|),$$

for $x_0-\lambda<x<x_0+\lambda$, where $k(x_0)>0$ and $\int_0^\lambda t^{-1}g^2(t)dt<\infty$.

F. V. Atkinson (Canberra)

ORDINARY DIFFERENTIAL EQUATIONS

See also 1899.

1806:

Brauer, Fred; and Sternberg, Shlomo. Local uniqueness, existence in the large, and the convergence of successive approximations. Amer. J. Math. **80** (1958), 421–430.

Let V be a continuous function defined for n-vectors x and real t satisfying: $V(x, t)\geqq 0$; $V(x, t)=0$ implies $x=0$; V has one-sided derivatives with respect to t and the components of x. The following uniqueness result is proved: Let ω be a continuous non-negative function defined for $r\geqq 0$, $0<t<a$. Suppose the only solution of $r'(t)=\omega(r, t)$, satisfying $r(0)=r'(0)=0$ on $0\leqq t\leqq\alpha$ for any α in $0<\alpha<a$, is the identically zero solution. Let f be a continuous n-vector on $0<t<a$, $V(x, t)<b$, and $V_t(x-y, t)+V_x\cdot[f(x, t)-f(y, t)]\leqq\omega[V(x-y, t), t]$ in this region. Then there is at most one solution of $x'(t)=f(x, t)$, $x(0)=0$, on $0\leqq t<a$. Here V_x is a gradient vector of V, and the dot indicates scalar product. The choice $V(x, t)=\sum_{i=1}^{n}|x_i|$ results in Kamke's general uniqueness theorem. A well-known example due to Müller is used to show that the hypotheses of the result above do not imply the convergence of the successive approximations. An example suggested by J. Dieudonné is exhibited for which the successive approximations converge for an arbitrary initial curve and yet the solution is not unique. By imposing an additional condition on V it is shown that the successive approximations converge. This result is sufficiently general to imply that the successive approximations converge under the hypotheses of Kamke's general uniqueness theorem. This removes a monotonicity assumption made in a result due to Levinson and the reviewer [J. Indian Math. Soc. **16** (1952), 75–81; MR **14**, 169]. A non-local existence result is given which generalizes one due to Wintner [Amer. J. Math. **68** (1946), 173–178; MR **7**, 297].

E. A. Coddington (Los Angeles, Calif.)

1807:

Eftimiu, C.; et Klarsfeld, S. Sur l'oscillateur linéaire relativiste. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **6** (1957), no. 13, 53–57. (Romanian. French and Russian summaries)

The author integrates the differential equations of the linear harmonic oscillator with a variable mass, as in relativity theory. The solution is expressed in elliptic integrals. The problem is not posed in a relativistically invariant form.

L. Markus (Minneapolis, Minn).

1808:

Amitsur, S. A. Commutative linear differential operators. Pacific J. Math. **8** (1958), 1–10.

The author studies the family $C(f)$ of all linear differential operators which commute with a given operator

$$f(D)=a_0D^n+a_1D^{n-1}+\cdots+a_n,\quad D=d/dx.$$

The first part of the paper gives a simplified proof of a result by Flanders that $C(f)$ is a commutative ring which is finitely generated over the ring of all polynomials in $f(D)$ with constant coefficients. In the second part the author obtains necessary and sufficient conditions for the existence of non-trivial operators which commute with $f(D)$. All proofs are purely algebraic.

O. Ore (New Haven, Conn.)

1809:

Kimura, Toshihusa. Generalization of a theorem of Malmquist. Sûgaku **8** (1956/57), 1–11. (Japanese)

An expository paper describing the author's investigations [Comment. Math. St. Paul. **2** (1954), 47–53; ibid. **4** (1955), 25–41; MR **15**, 705; **17**, 364] on the reduction of the differential equation

$$\text{(E)}\qquad x^{\sigma+1}\frac{dy}{dx}=\frac{P(x, y)}{Q(x, y)},$$

where σ is a non-negative integer, and P and Q are relatively prime polynomials in y whose coefficients are regular in a vicinity of $x=0$ and such that $P(0, y)\not\equiv 0$, $Q(0, y)\not\equiv 0$. Thus, based upon and developing the researches of M. Hukuhara [see the reference in the author's paper referred to above], the author proves the following extension of Malmquist's theorem [Acta Math. **36** (1912–13), 297–343]:

If (E) is not of Riccati's type, then it is possible to know, by purely algebraic operations, whether or not (E) admits a solution with movable branch points in the vicinity of $x=0$ ("movable" means that the location of the branch point of the solution varies with the initial condition of the solution). Moreover, if such solution with movable branch points exists, then (E) can be transformed into the equation of the form

$$x\frac{dz}{dx}=\frac{\lambda x^{\sigma+1}+\bar{P}(x, z)}{z+\bar{Q}(x, z)},$$

and $x=0$ is a logarithmic branch point of the corresponding solution. *K. Yosida* (Tokyo)

1810:

Saltykow, N. Problèmes d'intégration d'équations linéaires. Bull. Acad. Serbe Sci. Cl. Sci. Math. Nat. (N.S.) **10** (1956), no. 2, 49–87.

Betrachten wir die Differentialgleichung (1) $y''+ay'+by=f(x)$ mit konstanten Koeffizienten. Schreiben wir sie in der Form $(y'-ky)'+(a+k)(y'+by/(a+k))=f(x)$ und wählen die Konstante k so, dass $b/(a+k)=-k$, also $k^2+ak+b=0$ gilt. Dann bekommt man die allgemeine Lösung von (1), wenn man zwei lineare Differentialgleichungen erster Ordnung löst, nämlich $z'+(a+k)z=f(x)$ und $y'-ky=z$. Diese Idee wird von dem Verfasser verallgemeinert und zur Lösung der linearen Differentialgleichungen beliebiger Ordnung sowie auch der Systeme linearer Differentialgleichungen angewendet. Zum Ende wird ein neuer Beweis eines Liapounoffschen Stabilitätskriteriums geliefert. *M. Zlámal* (Brno)

1811:

Knobloch, Hans-Wilhelm. Zusammenhänge zwischen konvergenten und asymptotischen Entwicklungen bei Lösungen linearer Differentialsysteme vom Range eins. Math. Ann. **134** (1958), 260–288.

This paper treats a system of n linear differential equations of the form (1) $y'=yM$, where $M=\sum_{j=0}^{\infty}A_jx^{-j}$ converges for $|x|>k$, y represents a row vector with n components and M is an n by n matrix. It is presumed that equation (1) possesses n linear independent formal row vector solutions of the form

$$(2)\qquad u(x)=e^{\alpha x}x^{-\rho}\sum_{i=0}^{m}\log^i(x^{-1})\sum_{j=0}^{\infty}\alpha_{ij}x^{-j}$$

such that $|\alpha_{ij}|<j!c^j$. It is shown that to each $u(x)$ there corresponds a unique row vector $V(t)$ with the convergent representation $V(t)=\sum_{i=0}^{m}\log^i t\sum_{j=0}^{\infty}\tilde{\alpha}_{ij}t^{\rho+j-1}$ for sufficiently small t which under the prescribed normalization can be extended analytically along the real axis to $t=1$, such that in any sector within the half-plane $\mathrm{Re}(\alpha x)>0$, asymptotically $w(x)=\int_0^1 e^{\alpha(1-t)x}V(t)dt\sim\alpha^{-\rho}u(x)$.

Formal Laurent series $y(x)=\sum_{j=-\infty}^{\infty}b_jx^j$, where the sum of the positive powers of x always represent entire functions, are introduced. Then in terms of the $w(x)$ and $v(t)$ vectors, n linear independent formal Laurent series solutions $y_1(x),\cdots,y_n(x)$ are constructed. It is proved, among other things, that any formal Laurent series satisfying (1) is a linear combination of $y_1(x),\cdots,y_n(x)$. These results are then used to prove in a new way Hoheisel's theorem [J. Reine Angew. Math. **153** (1924), 228–244] that, if the general solution of (1) is single valued in the neighborhood of $x=\infty$, then the n formal solutions of type (2) all converge. Furthermore one of Ludwig Hopf's results [Math. Ann. **111** (1935), 678–712] obtained heuristically is extended and put on a rigorous basis. Roughly stated the result here is that, if system (1) has a particular solution $y(x)=\sum_{j=0}^{\infty}b_jx^j$ which is an entire function, then one can divide the complex x-plane into sectors S_λ in which $y(x)$ can be represented asymptotically by particular formal solutions $u_\lambda(x)$ of type (2) in such a way that the coefficients

$$b_j=(2\pi i)^{-1}\lim_{k\to\infty}\sum_{\lambda}\int_{\substack{|x|=k\\ x\in S_\lambda}}x^{-j-1}u_\lambda(x)dx,$$

where the integration is made termwise, next limits are taken termwise, and finally the summation is carried out. *H. L. Turrittin* (Minneapolis, Minn.)

1812:

Tereščenko, N. I. On the solution of some systems of linear differential equations with singular points. Ukrain. Mat. Ž. **10** (1958), no. 2, 220–223. (Russian)

1813:

Tereščenko, N. I. On asymptotic logarithmic solutions of linear homogeneous differential equations. Ukrain. Mat. Ž. **10** (1958), no. 1, 82–83. (Russian)

1814:

Borůvka, O. Sur la transformation des intégrales des équations différentielles linéaires ordinaires du second ordre. Ann. Mat. Pura Appl. (4) **41** (1956), 325–342.

Given a function $X(t)$ with a non-vanishing derivative X', define $\{X,t\}=\frac{1}{2}X'''X'^{-1}-\frac{3}{4}X''^2X'^{-2}$; similarly for functions of T, the derivatives being indicated by dots. Let q, Q be two continuous functions and consider equations (b): $-\{X,t\}+Q(X)X'^2=q(t)$, (B): $-\{x,T\}+q(x)\dot{x}^2=Q(T)$, (b′): $-\{X,t\}+q(X)X'^2=q(t)$, (B′): $-\{x,T\}+Q(x)\dot{x}^2=Q(T)$. The following results are typical: 1. If X is an integral of (b) its inverse function is an integral of (B); 2. If $X, y, \bar{X}, \bar{y}$ are integrals of (b), (B), (b′), (B′), respectively, the composite functions $X\bar{X}$, $\bar{y}X$, $y\bar{y}$, $\bar{X}y$, yX, Xy are solutions of (b), (b), (B), (B), (b′), (B′), respectively; 3. Let X be an integral of (b) and U an integral of (A): $Y''=Q(T)Y$; then (13): $u=U(X)\cdot X'^{-\frac{1}{2}}$ is an integral of (a): $\ddot{y}=q(t)y$; and conversely (with certain restrictions which we do not reproduce explicitly) if u, U are integrals of (a), (A), there is an integral X of (b) such that (13) holds.

J. L. Massera (Zbl **72**, 89)

1815:

Wintner, Aurel. On an inequality of Liapounoff. Quart. Appl. Math. **16** (1958), 175–178.

Let $p(t)$ be continuous and non-negative on $[a,b]$ and suppose that $x''+p(t)x=0$ does not have a solution which is a non-vanishing constant on some t-interval ($p(t)>0$ is sufficient for this latter condition). A closed t-interval $[c,d]$ is called a primitive interval of $x(t)$ if $x(t)\not\equiv 0$ and $x'(t)\not\equiv 0$ on the open interval (c,d). It is shown that, under the above hypotheses on $p(t)$, if $[a,b]$ consists of exactly n primitive intervals of some solution $x(t)\not\equiv 0$ of $x''+p(t)x=0$ then $\int_a^b p(t)dt>n^2/(b-a)$. {The inequality sign of formula line (4) is a misprint and should be reversed.}

C. R. Putnam (Lafayette, Ind.)

1816:

Švec, Marko. Sur une propriété des intégrales de l'équation $y^{(n)}+Q(x)y=0$, $n=3, 4$. Czechoslovak Math. J. **7(82)** (1957), 450–462. (Russian summary)

The author considers the real differential equations a) $y^{(4)}+Q(x)y=0$ and b) $y^{(3)}+Q(x)y=0$, where $Q(x)$ is a continuous, non-negative function on $-\infty<x<\infty$, and the set of zeros of $Q(x)$ has no interior. Theorem: If one solution of (a) is oscillatory (at $+\infty$), then so is every solution. Theorem: If one solution of (b) is oscillatory then each solution is either oscillatory or is nowhere zero on $-\infty<x<\infty$. For the solutions which are nowhere zero, $y(x)$, $y'(x)$, $y''(x)$ are monotone functions, and $\operatorname{sgn} y(x)=\operatorname{sgn} y''(x)\neq\operatorname{sgn} y'(x)$ and $\lim_{x\to+\infty}y(x)=\lim_{x\to\infty}y'(x)=\lim_{x\to\infty}y''(x)=0$. The analogous statements are false for equations of higher order, as shown by $y^{(n)}+kx^{-n}y=0$, for some small $k>0$.

L. Markus (Minneapolis, Minn.)

1817:

Kemp, R. R. D. A singular boundary value problem for a non-self-adjoint differential operator. Canad. J. Math. **10** (1958), 447–462.

The subject of this paper is the eigenfunction expansion theory associated with the operator $L=-d^2/dx^2+g(x)$, where $g(x)$ is a complex-valued function such that $(x^2+1)^{\frac{1}{2}}|g(x)|$ is integrable over $(-\infty, \infty)$. It is found that there is a function $W(s)$, analytic for $\text{Im } s>0$ and continuous for $\text{Im } s\geqq 0$, such that the squares of its zeros in $\text{Im } s>0$ constitute a bounded set which is the point spectrum of L. The continuous spectrum is the set of all real non-negative numbers. An expansion theorem is obtained by a method involving contour integration.

E. C. Titchmarsh (Oxford)

1818:

Briš, N. I. On the first boundary problem for a linear equation of fourth order with a small parameter in the term with the highest derivative. Minsk. Gos. Ped. Inst. A. M. Gor'k. Uč. Zap. **5** (1956), 3–13. (Russian)

The problem discussed is the behavior of the solutions on $[a, b]$ of $\varepsilon y^{(4)}-A(x)y''+B(x)y'+C(x)y=f(x)$, $y_\varepsilon(a)=y_\varepsilon(b)=y_\varepsilon'(a)=y_\varepsilon'(b)=0$, where ε is a small positive parameter. Under certain restrictions, notably that A, B, C, f be respectively of classes C^4, C^3, C^2, C^2 on $[a, b]$, and with u the solution of the degenerate equation (with $\varepsilon=0$) such that $u(a)=u(b)=0$, setting also $\varepsilon^{\frac{1}{2}}=\eta$, there is obtained a representation of the form

$$y_\varepsilon(x)=u(x)+\eta\{h_1(x)e^{-\theta_1(x)/\eta}+h_2(x)e^{-\theta_2(x)/\eta}\}+w(x, \varepsilon),$$
$$w=O(\eta).$$

[References: Briš, Dokl. Akad. Nauk SSSR **95** (1954), 429–432; MR **16**, 251; Picone, Math. Z. **28** (1928), 519–555.] *S. Lefschetz* (Princeton, N.J.)

1819:

Minorsky, Nicolas. Sur l'excitation paramétrique. C. R. Acad. Sci. Paris **247** (1958), 406–408.

Using the stroboscopic method, the author discusses the existence and stability of periodic solutions of the equation

$$x''+bx'+x+(a-cx^2)x\cos 2t+ex^3=0$$

where a, b, c and e are small positive real numbers. This is a continuation of his work on equations of this type [same C.R. **232** (1951), 1060–1062; 2179–2180; MR **12**, 611; **13**, 38]. *J. K. Hale* (Baltimore, Md.)

1820:

Turrittin, H. L.; and Culmer, W. J. A. A peculiar periodic solution of a modified Duffing's equation. Ann. Mat. Pura Appl. (4) **44** (1957), 23–33.

Consider the equation

$$(1)\qquad x''+hx'+x+x^3=B\cos\omega t+(A^3\cos\omega t)/4,$$

where $B=A-A\omega^2+3A^4/4$. Since the Fourier series of the forcing function contains only odd harmonics, one might expect that a periodic solution of (1) also would contain only odd harmonics. In this paper, the authors use Poincaré's perturbation method, together with some special properties of the solutions of Mathieu's equation, to show that there exists a periodic solution of (1) whose Fourier series contains both odd and even harmonics for special values of the parameters h, A and ω, with h and $A\omega^{-1}$ sufficiently small. {Reviewer's note: If $x=A\cos\omega t+hy$, $\omega t=u$, then (1) is equivalent to a second order equation in y which is weakly nonlinear if h and $A\omega^{-1}$ are small. By choosing h and $A\omega^{-1}$ to be convenient functions of a common parameter, b, one can prove the above result by using a method of successive approximations developed by L. Cesari, J. K. Hale and R. A. Gambill [see, e.g., Gambill and Hale, J. Rational Mech. Anal. **5** (1956), 353–394; MR **17**, 1086] which does not require any special properties of the Mathieu equation. This method has already been applied to numerous equations similar to (1).} *J. K. Hale* (Baltimore, Md.)

1821:

Wintner, Aurel. On applications of the Schwarzian derivative in the real domain. Boll. Un. Mat. Ital. (3) **12** (1957), 394–400.

The author reformulates some earlier non-oscillation criteria for the equation $y''+f(t)y=0$ in terms of the equivalent third-order equation $\{y, t\}=2f(t)$, where $\{y, t\}$ is the Schwarzian derivative.

Z. Nehari (Pittsburgh, Pa.)

1822:

Volosov, V. M. Solutions of second order non-linear differential equations with slowly varying coefficients. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1149–1152. (Russian)

The problem studied here is similar to one previously considered by the same author [same Dokl. (N.S.) **106** (1956), 7–10; MR **18**, 41]. The equation is

$$d[m(\varepsilon t)\dot{x}]/dt+q(\varepsilon t, x)+\varepsilon\varphi_1(\varepsilon t, x, \dot{x})+\varepsilon^2\varphi_2(\varepsilon t, x, \dot{x})=0,$$

ε small. The results are also similar but the behavior of the solutions is described with a higher degree of approximation (up to terms in ε). The equations and formulae derived are extremely complicated. *J. L. Massera* (Montevideo)

1823:

Volosov, V. M. Non-linear oscillations with one degree of freedom of a system with slowly varying parameters. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 927–930. (Russian)

In previous notes [see review above and reference therein; also same Dokl. **115** (1957), 20–22; MR **19**, 276] the author had dealt with the oscillatory solutions of

$$\ddot{x}+Q(\varepsilon t, x)+\varepsilon f(\varepsilon, \varepsilon t, x, \dot{x})=0$$

(ε very small). The frequency and amplitude were dealt with over an interval of the order $1/\varepsilon$. In the present note expressions from which one may solve for x and $\dot{x}$ (to order ε) in terms of initial conditions x_0, $\dot{x}_0$ are given. [Additional references: Mitropol'skiĭ, Transient processes in non-linear oscillatory systems, Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1955; Bogolyubov and Mitropol'skiĭ, Asymptotic methods in the theory of non-linear oscillations, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955; MR **17**, 135, 368.] *S. Lefschetz* (Princeton, N.J.)

1824:

Bihari, I. Oscillation and monotonity theorems concerning non-linear differential equations of the second order. Acta Math. Acad. Sci. Hungar. **9** (1958), 83–104.

This paper discusses a generalization of a theorem of W. E. Milne [Bull. Amer. Math. Soc. **28** (1922), 102–104] concerning the non-linear differential equation $y''+\phi(x)f(y)=0$, where $\phi(x)$ is a positive, continuous, increasing, and bounded function for $x\geqq a$, and $f(y)$ is an increasing odd function with $f(y)\in \text{Lip}(1)$ for $|y|\leqq b$. It

also discusses extensions of certain comparison theorems of Sturm. The following results are established.

Theorem 1: Suppose that in the non-linear equation $y''+\phi(x)f(y)h(y')=0$, (i) $\phi(x)>0$ is continuous, increasing and bounded for $x\geqq a$, (ii) $f(y)$ is an odd non-decreasing function and $f(y)\in \mathrm{Lip}(1)$ for $|y|\leqq b$, (iii) $h(u)>0$ is non-decreasing for $u\leqq 0$ and non-increasing for $u\geqq 0$, and $h(u)\in \mathrm{Lip}(1)$ for all u; then the equation has a unique solution with the initial conditions $y(a)=\eta$, $y'(a)=0$ $(0<|\eta|\leqq b)$; this solution exists for all $x\geqq a$, oscillates an infinity of times, the amplitudes decrease but do not approach zero, and y' remains bounded. A number of corollaries are also given.

Theorem 2: Let $f(x, y, u)$ be defined for $x\geqq a$ and for arbitrary y and u, with the following properties: (i) $f(x, y, u)$ is continuous and $\operatorname{sgn} f(x, y, u)=-\operatorname{sgn} y$; (ii) $f_x(x, y, u)$, $f_y(x, y, u)$, $f_u(x, y, u)$ exist and are continuous; (iii) $\operatorname{sgn} f_x=-\operatorname{sgn} y$; (iv) $f_y(x, y, u)<0$ and $f_u(x, y, u)>0$ if $\operatorname{sgn} u=\operatorname{sgn} y\neq 0$, $=0$ if $y=0$, <0 if $\operatorname{sgn} u=-\operatorname{sgn} y\neq 0$ $(x\geqq a)$; (v) let $f(x, y, u)$ satisfy the Nagumo condition, i.e., let a continuous positive function $\Phi(u)$ $(u\geqq 0)$ exist satisfying the conditions $|f(x, y, u)|\leqq\Phi(|u|)$, $\int_0^\infty u/\Phi(u)du=+\infty$; then there exists for $x\geqq a$ a unique solution of the general form of the explicit equation $y''=f(x, y, y')$, with the initial conditions $y(a)=\eta\neq 0$, $y'(a)=0$, and oscillating an infinity of times.

Theorem 3: If $h(u)$ in Theorem 1 is an even function, then, considering equal values of y on a half-wave (in the graph of the solution), the slope $|y'|$ is greater to the right of the maximum than to the left of it. Consequently, if the abscissae of the points where the half-wave considered meets the x-axis, and of the maximum, are respectively x_1, x_2, x_m, then the symmetrical of the left-hand "quarter" wave to the ordinate at x_m will lie entirely above the right-hand "quarter" wave; hence its area is greater, and finally, $x_m-x_1>x_2-x_m$.

Theorem 4 shows that under the conditions of Theorem 3, except that the restriction that $h(u)$ is even is omitted, the areas of successive half-waves form a steadily decreasing sequence.

Theorem 5 is concerned with the extension of one of Sturm's comparison theorems to the equations $y''+\phi(x)f(y)h(y')=0$, $\eta''+\psi(x)f(\eta)h(\eta')=0$.

Further discussion of the areas of successive half-waves of the solution of $y''+\phi(x)f(y)h(y')=0$, assuming the conditions of Theorem 5, is given in Theorem 6 and the remaining sections. *R. G. Cooke* (London)

1825:

Atrašenok, P. V. Some questions in the theory of stability of motion. Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him. **9** (1954), no. 8, 79–106. (Russian)

1826:

Halanay, A. Quelques observations sur la stabilité asymptotique. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 9, 31–38. (Romanian. Russian and French summaries)

Consider the complex differential system

$$\frac{dx}{dt}=A(t)x+F(x, t). \tag{1}$$

Assume, for $t>0$, $F(x, t)$ satisfies a Lipschitz condition in x for a suitably small Lipschitz constant. Let $\lambda_M(t)$ be the largest characteristic root of the Hermitian matrix $\frac{1}{2}(A+\bar{A}')$. Let $G(t, \tau)=\exp\int_\tau^t \lambda_M(u)du$. If $G(t, 0)$ and $\int_0^t G(t, \tau)d\tau$ are bounded for $t>0$, then the solutions of (1) all tend asymptotically to the origin as $t\to+\infty$. This theorem, and similar results, are proved using Perron's reduction of the coefficient matrix $A(t)$ to a triangular matrix. *L. Markus* (Minneapolis, Minn.)

1827:

Klyamko, È. I. Some applications of the method of Čaplygin to the approximate solution of differential equations with a retarded argument. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 305–312. (Russian)

Consider the scalar equation

$$y'-F(x, y(x), y(x-\tau(x)))=0, \tag{1}$$

defined in $[0, H]$, with a given initial condition in $[\alpha, 0]$, $\alpha<0<H$; it is assumed that $F(p, q, s)\in C^2$, $F_s\geqq 0$, $\tau\geqq 0$ continuous. Then if $z(x)$ is a solution of the equation with a second member $\omega(x)\geqq 0$ and with the same initial condition, $z\geqq y$ in $[0, H]$. Under certain still more restrictive assumptions on F an extension of Čaplygin's method follows. Let in particular (1) be linear; by means of a preliminary change of variables, the equation is reduced to $y'-a(x)y(x-\tau(x))=f(x)$, $a\geqq 0$; assume that $z(x)$ is such that $z'-a(x)z(x-\tau(x))=f(x)+\omega(x)$, $\omega\geqq 0$, whence $z\geqq y$; define $z_1(x)=\int_0^x \omega(t)dt$ and then the successive approximations $z_n(x)=\int_0^x a(t)z_{n-1}(t-\omega(t))dt$; it is shown that $y=z-\sum_{n=1}^\infty z_n$. For second order equations $y''(x)+a(x)y'(x)+b(x)y(x)=f(x)+c(x)y(x-\tau(x))$, $c\geqq 0$, a similar process may be applied, but the computation of the successive approximations requires the solution of a linear second-order differential equation with variable coefficients; if however $b\geqq 0$ it is possible to introduce modifications whereby only quadratures are needed at each step. *J. L. Massera* (Montevideo)

PARTIAL DIFFERENTIAL EQUATIONS

See also 1912, 1920, 1925, 2016, 2017, 2018, 2088, 2161.

1828:

Il'in, V. A. On convergence of expansions in eigenfunctions of the Laplace operator. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1(79), 87–180. (Russian)

The paper discusses the expansion of an arbitrary function $f(Q)$ with respect to the eigenfunctions of the equation

$$\Delta u+\lambda u=0$$

in an n-dimensional domain g under homogeneous boundary conditions of one of the three types: (I) $u=0$; (II) $\partial u/\partial\nu=0$; (III) $\partial u/\partial\nu+h(S)u=0$; in (II) and (III) ν denotes the outward normal to the boundary, and in (III) it is assumed that $h(S)\geqq 0$.

The sequence of eigenvalues is denoted by $\{\lambda_i\}$ and the corresponding eigenfunctions by $u_i(Q)$.

A function f is said to belong to the Sobolev class $W_p^{(l)}(g)$ if f is summable and has generalized (distribution-theoretical) derivatives up to the lth order, all belonging to the class $L^p(g)$.

A function f is said to satisfy the boundary conditions in the generalized sense if: (i) the boundary Γ of g is sectionally smooth; (ii) g' denotes one of a family of domains interior to g and converging to g in such a way that $\Gamma'\to\Gamma$ and the normals at points of Γ' tend to the normals at the corresponding points of Γ; (iii) in case (I),

$$\lim_{\Gamma'\to\Gamma}\int_\Gamma f\Psi dS=0$$

for all $\Psi \in W_2^{(1)}(g)$; (iv) in case (II) or (III),

$$\lim_{\Gamma' \to \Gamma} \int_{\Gamma'} \left[\frac{\partial f}{\partial \nu} + h(S) f \right] \Psi dS = 0$$

for all $\Psi \in W_2^{(1)}(g)$, $h(S)$ being prolonged in a continuous way to interior points of g.

The paper begins with a survey of the literature on the convergence of the expansion

$$f(Q) = \sum_{i=1}^{\infty} f_i u_i(Q)$$

of an arbitrary function $f(Q)$, including some recent results of the author on its absolute convergence, to be published in detail elsewhere. The author then states his principal result, which is as follows.

Let g be an arbitrary N-dimensional domain, $N>1$, whose boundary is such that Green's formula can be applied to the eigenfunctions $u_i(Q)$. Suppose that

(i) $$f \in W_p^{[\frac{1}{2}N]}(g), \quad p > N/[\tfrac{1}{2}N];$$

(ii) the functions $f, \Delta f, \Delta^2 f, \cdots, \Delta^k f$ satisfy the appropriate homogeneous boundary condition in the generalized sense, where $k=[(N-2)/4]$ in case (I), $k=[(N-4)/4]$ in case (II) or (III). Then the eigenfunction expansion

$$\sum_{i=1}^{\infty} f_i u_i(Q)$$

of f, arranged in order of increasing λ_i, is uniformly convergent to f in every domain g' strictly interior to g.

If N is even, then the series

$$\sum_{i=1}^{\infty} f_i u_i(Q) \lambda_i^{\alpha}$$

is also uniformly convergent in every strictly interior g', provided that

$$\alpha < \frac{(p-2)N}{4p} \left(2<p<\frac{2N}{N-1}\right), \quad \alpha < \frac{1}{4} \left(p > \frac{2N}{N-1}\right),$$

and we have

$$r_m(Q) = \sum_{i=m}^{\infty} f_i u_i(Q) = o(\lambda_m^{-\alpha}).$$

The author draws special attention to the situation when $N=2$ or $N=3$; here no conditions are imposed on the second derivatives of f and, in cases (II) and (III), f is not even required to satisfy any boundary conditions.

The results are generalized to cover sectionally smooth functions in the case $N=2$; and also, for general N, to cover functions having a singularity of type $\log r$ or $r^{-\alpha}$, with $0<\alpha<1$, at an interior point P of g. As a consequence of the last result, the bilinear expansion

$$K_1(P, Q) = \sum_{i=1}^{\infty} \frac{u_i(P) u_i(Q)}{\lambda_i}$$

for the Green's function is convergent when $P \neq Q$ for an arbitrary bounded two-dimensional domain possessing a Green's function.

The proof of the main theorem occupies the bulk of the paper, and is too long and complicated even to be outlined here.

Examples are given to show that all the constants in the main theorem are best possible, i.e. the order $[\frac{1}{2}N]$, the lower bound $N/[\frac{1}{2}N]$ on the exponent p, and the order k cannot be reduced. The only remaining doubt is in cases (II) and (III) when N is of the form $4n+2$; here it may be possible to reduce k by 1.

Finally, it is shown that p can be reduced if f is known to be smoother; more precisely, the theorem still holds if

$$f \in W_p^{[\frac{1}{2}N]+m}(g), \quad p > \frac{N}{[\frac{1}{2}N]+m}, \quad 0 \leqq m \leqq [\tfrac{1}{2}N+\tfrac{1}{2}];$$

here again the lower limit on p is best possible.

F. Smithies (Cambridge, England)

1829:

Il'in, V. A. On expansion of functions with singularities in a conditionally convergent series of eigenfunctions. Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 49–80. (Russian)

As in a previous paper of the author [Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 21–24; MR **18**, 402], the functions to be expanded have singularities of the form $r^{-\alpha}$ or $\log r$, and the expansion is in terms of the eigen-functions u_i of $(\Delta+\lambda)u=0$, numbered in order of increasing λ_i. It is emphasised that the expansion need not be absolutely convergent anywhere. Whereas the previous paper relied on an asymptotic formula for λ_i and on results of B. M. Levitan [Mat. Sbornik N.S. **35(77)** (1954), 267–316; MR **17**, 158], the author now gives independent derivations of asymptotic formulae for $\sum_{\lambda_i \leq \mu} u_i(P) u_i(Q)$ and related sums. The present method is claimed to give greater generality in regard to the boundary of the region.

F. V. Atkinson (Canberra)

1830:

Bergendal, Gunnar. On spectral functions belonging to an elliptic differential operator with variable coefficients. Math. Scand. **5** (1957), 241–254.

The author establishes some asymptotic properties of spectral functions belonging to semi-bounded, self-adjoint extensions of elliptic operators. Let $a(x, D)$ be a symmetric elliptic partial differential operator of order m with sufficiently differentiable coefficients in an open subset S of R^n. If $C_0^m(S)$ is the set of m times continuously differentiable functions vanishing outside compact subsets of S, let A_1 and A_2 be any semi-bounded, self-adjoint extensions of a considered as an operator from $C_0^m(S)$ to $L^2(S)$. Then for large λ

$$I^k(e_1(\lambda, x, y) - e_2(\lambda, x, y)) = O(\lambda^{(n+1-k)/m})$$

uniformly on compact subsets of $S \times S$, where

$$I^k \sigma(\lambda) = \Gamma(k)^{-1} (\lambda - \lambda_0)^{1-k} \int_{\lambda_0}^{\lambda} (\lambda - \mu)^{k-1} d\sigma(\mu),$$

e_1 and e_2 are the spectral functions corresponding to A_1 and A_2, respectively, $k \geqq 1$, and $\lambda_0 < \min(A_1, A_2)$.

The proof consists of deriving asymptotic estimates for the Green's function for the parabolic operator $a - \partial/\partial t$. The result then follows via a Tauberian argument.

For the constant coefficient case, Gårding [Kungl. Fysiogr. Sällsk. i Lund Förh. **24** (1954), no. 21; MR **17**, 158] had previously proved the sharper result

$$I^k(e(\lambda, x, y) - e_0(\lambda, x, y)) = O(\lambda^{(n-k)/m})$$

uniformly on compact subsets of $S \times S$, where

$$e_0(\lambda, x, y) = (2\pi)^{-n} \int_{a(\xi) < \lambda} e^{-i(x-y)\xi} d\xi$$

is the spectral function corresponding to the unique extension of a in R^n.

The author also includes a new proof of another result of Gårding [ibid.]. *M. Schechter* (New York, N.Y.)

1831:

Mangeron, D. Connections between solutions of different boundary value problems of higher order. Bul. Inst. Politehn. Iaşi (N.S.) 3 (1957), no. 1–2, 39–42. (Romanian. Russian and English summaries)

A short note on considerations about partial differential equations of higher order given by the author elsewhere [Bul. Politehn. Gh. Asachi Iaşi 1 (1946), 295–303; MR 8, 466]. *O. Bottema* (Delft)

1832:

Hartman, Philip. On integrating factors and on conformal mappings. Trans. Amer. Math. Soc. 87 (1958), 387–406.

The main results of this paper deal with the existence of (local) integrating factors T for a linear differential form $\omega=P(x, y)dx+Q(x, y)dy$ in the two cases: (1) P and Q are real valued and $P^2+Q^2\neq 0$; (2) P and Q are complex valued and $\mathrm{Im}(P\bar{Q})\neq 0$. In both cases, the existence of functions T and w are established such that $\omega=Tdw$. In case (1), the solution of the problem depends on the theory of ordinary differential equations. If P and Q are continuous on $x^2+y^2\leq 1$, and if there exists a continuous function f satisfying $\int_J Pdx+Qdy=\iint_E f dxdy$ for every rectangle E with boundary J in $x^2+y^2\leq 1$, then a continuous integrating factor exists on a sufficiently small neighborhood of the origin. In case (2), the solution depends on an elliptic system of partial differential equations. In this case a theorem analogous to that stated above for case (1) obtains. This result may be restated in an equivalent form which gives conditions that a positive definite binary Riemannian metric $ds^2=Edx^2+2Fdxdy+Gdy^2$ be reduced to the form $ds^2=|T|^2(du^2+dv^2)$ by a mapping of class C'. *A. Fialkow* (Brooklyn, N.Y.)

1833:

Drăgan, I. Sur les équations aux dérivées partielles du second ordre $F(x, y, z, p, q, r, s, t)=0$ intégrables par la méthode de Darboux. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. 7 (1956), no. 1, 71–117. (Romanian. Russian and French summaries)

Goursat, in his 1896–98 "Leçons sur l'intégration des équations aux dérivées partielles du second ordre" (vol. 2, Chapter VIII, sec. 183), proposed the problem of determining all equations for which Darboux's method can be applied successfully to the systems of characteristics; and he completely solved the problem (Chap. VII, sec. 166) in the special case that the two systems of characteristics of the partial differential equation coincide. Later, Vessiot [J. Math. Pures Appl. (9) 21 (1942), 1–66; MR 5, 67; and earlier papers] studied some equations with distinct characteristic systems which are integrable by Darboux's method. In the present paper, the author determines the forms of all partial differential equations with distinct characteristic systems for which Darboux's method can be applied successfully to both systems of characteristics.

A. Erdélyi (Pasadena, Calif.)

1834:

Kostyukovič, E. H. On convergence of the straight-line method applied in various ways to solving the first boundary problem for differential equations of elliptic type. Grodnen. Gos. Ped. Inst. Uč. Zap. Ser. Mat. 2 (1957), 60–78. (Russian)

1835:

Fishel, B. Boundary-value problems for second-order, formally self-adjoint, elliptic differential equations. J. London Math. Soc. 33 (1958), 62–70.

Quelques compléments sur les prolongements self-adjoints d'opérateurs non bornés.

Applications aux prolongements de $-\Delta+p$, sur un ouvert Ω de R^n, p fonction réelle mesurable et bornée sur Ω. Par exemple, si Ω est un ouvert borné, il y a des extensions self-adjointes de spectre non discret.

J. L. Lions (Nancy)

1836:

Miranker, Willard L. The asymptotic theory of solutions of $\Delta u+k^2u=0$. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-21** (1957), i+48 pp.

The first, third and fourth sections of this report are essentially identical, in this order, with the following papers by the same author: J. Math. Mech. 6 (1957), 847–858 [MR 19, 1056]; Ann. of Math. (2) 67 (1958), 72–82 [MR 19, 964]; Arch. Rational Mech. Anal. 1 (1957), 139–153 [see the following review]. In section 2 the results of the first section are extended to solutions of the equation

$$\Delta v+k^2(x_1, x_2, x_3)v=0,$$

under the assumption that k approaches a finite limit at infinity. It is shown that the growth estimate of section I remains valid if u is defined and bounded in the complement of a bounded domain. From this result various uniqueness and representation theorems for the solutions of this equation are deduced.

W. Wasow (Madison, Wis.)

1837:

Miranker, W. L. Parametric theory of $\Delta u+k^2u=0$. Arch. Rational Mech. Anal. 1 (1957), 139–153.

Let $u(x, y)$ be a solution of $\Delta u+k^2u=0$ in the exterior of a bounded or unbounded domain D and assume that $u(x, y)$ satisfies a condition of the form

$$(1)\qquad u=u_0(x, y)=\exp[ik\phi(x, y)]\sum_{n=0}^{M}\frac{a_n(x, y)}{(ik)^n}$$

on the boundary B of D, as well as the radiation condition

$$\lim_{R\to\infty}\int_{C_R}\left|\frac{\partial u}{\partial r}-iku\right|^2 ds=0,$$

where C_R is the portion outside D of a circle of radius R with fixed center. Following a procedure suggested by G. D. Birkhoff [Bull. Amer. Math. Soc. 39 (1933), 681–700], a series of the form

$$(2)\qquad u^\infty=\exp[ik\psi(x, y)]\sum_{n=0}^{\infty}\frac{\sigma_n(x, y)}{(ik)^n}$$

is constructed which satisfies the differential equation in the formal sense and reduces to (1) on B. The principal aim of the paper is to study the degree to which the Nth partial sum u^N of (2) approximates the solution $u(x, y)$, as $k\to\infty$. To this end the author imposes a certain regularity condition on the problem, which is too involved to be stated in this review, but which is satisfied in many problems of physical interest. The most precise result is obtained when B is a simple closed curve, and $\mathrm{Im}\,k>0$. In that case $u^N=u+O(k^{-N+19/15})/(\mathrm{Im}\,k)^2$. If only the boundedness of D is assumed u^N is shown to be of the form $O(k^{-N-2/5})+u^*$, where u^* is a solution of the same differential equation as u with boundary values that are also $O(k^{-N-2/5})$. This result, and some additional ones,

remain valid when B is infinite, provided the problem is again "regular", and if the boundary values and their normal derivatives tend sufficiently fast to zero at infinity.

In his proofs the author makes extensive use of his results in the paper reviewed above.

W. Wasow (Madison, Wis.)

1838:

Werner, Helmut. Das Problem von Douglas für Flächen konstanter mittlerer Krümmung. Math. Ann. 133 (1957), 303–319.

L'A. studia il problema di Douglas per le superficie con curvatura media costante, mediante procedimenti variazionali che si riallacciano a quelli noti per il problema delle superficie minime [v. ad es. R. Courant, Dirichlet's principle..., Interscience, New York, 1950; MR 12, 90] e a quelli sviluppati da E. Heinz per lo stesso problema [Math. Ann. 127 (1954), 258–287; MR 16, 1115]. Il risultato pricipale cui arriva è il seguente. Siano date nel cubo unitario n curve di Jordan chiuse $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ tali che: I) non sia vuota la classe A dei vettori $\mathfrak{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ definiti in un dominio D n-volte connesso (eventualmente variabile con $\mathfrak{x}(u, v)$) delimitato dalla circonferenza K_1 unitaria del piano (u, v) e da altre $n-1$ circonferenze $K_2, \cdots, K_n$ interne ad essa, continui in $\bar{D}$, differenziabili con continuità in D, con integrale di Dirichlet finito, con $|\mathfrak{x}(u, v)| \leq 1$ e tali che K_i $(i=1, \cdots, n)$ venga da $\mathfrak{x}(u, v)$ rappresentato in modo "debolmente monotono" su Γ_i; II) esista in A una successione di vettori $\mathfrak{x}_n(u, v)$ $((u, v) \in D_n)$ minimizzante per l'integrale

$$E(\mathfrak{x}) = \iint_D \{\mathfrak{x}_u^2 + \mathfrak{x}_v^2 + \tfrac{1}{2}H\mathfrak{x} \circ (\mathfrak{x}_u \times \mathfrak{x}_v)\} du dv$$

(H costante tale che $|H| < \frac{1}{2}$, $\circ$ prodotto scalare, $\times$ prodotto vettoriale) tale che si possa determinare un numero $\delta > 0$ per cui ogni curva γ la cui immagine $\mathfrak{x}(\gamma)$ abbia diametro minore di δ, si possa contrarre con continuità a un punto. Nelle ipotesi I) e II) esiste il minimo di $E(\mathfrak{x})$ in A e il vettore minimante $\mathfrak{x}(u, v)$ verifica di più il sistema di equazioni

$$\Delta \mathfrak{x} = 2H(\mathfrak{x}_u \times \mathfrak{x}_v), \quad \mathfrak{x}_u^2 = \mathfrak{x}_v^2, \quad \mathfrak{x}_u \circ \mathfrak{x}_v = 0.$$

E. Magenes (Genova)

1839:

Copson, Edward T. Un théorème d'unicité pour l'équation des ondes à une dimension. C. R. Acad. Sci. Paris 246 (1958), 2562–2564.

If $u(x, t)$ satisfies the conditions: (A) u, u_t, u_x are continuous in the closed rectangle Δ: $0 \leq x \leq l$, $0 \leq t \leq T$; (B) u_{tt}, u_{xx}, u_{tx}, u_{xt} exist and are bounded in the open rectangle, and $u_{xt} = u_{tx}$ there; (C) $u(x, 0) = u_t(x, 0) = 0$, $0 \leq x \leq l$; (D) $u_x u_t = 0$ for $x = 0$, l and $0 \leq t \leq T$; (E) $-u_{tt} + u_{xx} = 0$ in the open rectangle; then $u(x, t) = 0$ in Δ.

R. McKelvey (Boulder, Colo.)

1840:

Methée, Pierre-Denis. L'équation des ondes avec second membre invariant. Comment. Math. Helv. 32 (1957), 153–164.

In a previous paper [Comment. Math. Helv. 28 (1954), 225–269; MR 16, 255] the author determined all distributions T which are invariant under proper rotations of the Lorentz group and satisfy $(\Box + k)T = 0$; here k is a complex constant and $\Box = \partial^2/\partial x_n^2 - \sum_1^{n-1} \partial^2/\partial x_i^2$. In this paper the author determines an invariant distribution T which satisfies $(\Box + k)T = Z$, where Z is a given invariant distribution.

The main point in the proof is the author's representation [in paper cited above] of invariant distributions. The construction of T is then reduced, essentially, to the solution of ordinary differential equations.

The results of this paper are also extended to the case where $\Box$ is replaced by an ultrahyperbolic operator.

L. Ehrenpreis (Waltham, Mass.)

1841:

***Sauer, Robert. Anfangswertprobleme bei partiellen Differentialgleichungen. 2te Aufl.** Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Bd. 62. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. xvi+284 pp. DM 41.00.

The first edition (1952) was reviewed in MR 14, 559; it has been improved and corrected in a number of places, for instance by inclusion of a chapter on differential equations of mixed type. There is a new chapter on Laurent Schwartz' theory of distributions which also introduces, briefly, the pseudo-functions of Marcel Riesz and the theory and applications of Laplace transforms.

1842:

Pini, Bruno. Sul primo problema di valori al contorno per le equazioni paraboliche lineari. Riv. Mat. Univ. Parma 6 (1955), 215–237.

In a domain in the (x, y) plane bounded by a segment on $y = y_1$, a segment on $y = y_2$ and two curves joining their extremities $x = X_1(y)$, $x = X_2(y)$, $y_1 \leq y \leq y_2$, $X_1(y) < X_2(y)$, the author considers the boundary value problem for the heat equation $L_0 u \equiv u_{xx} - u_y = 0$ and a more general parabolic equation $Lu \equiv u_{xx} - u_y + au_x + bu = 0$ with $u = \phi$ prescribed on the boundary except on the top segment. For non-smooth boundaries and continuous ϕ he defines a notion of generalized solution analogous to that of Wiener's for the Laplace equation. This is the limit of solutions of analogous boundary value problems for a monotonic increasing sequence of interior domains. {The existence of the limit can be established under more general conditions than those of the paper if one uses the Schauder estimates for parabolic equations; see C. Ciliberto, Ricerche Mat. 3 (1954), 40–75 [MR 16, 139]; A. Friedman, J. Math. Mech. 7 (1958), 393–417.}

A boundary point is called regular if for any continuous boundary values ϕ the generalized solution actually takes on the prescribed value there continuously. The main result of the paper is that (under certain smoothness hypotheses on the coefficients) a boundary point is regular for L if and only if it is regular for L_0. (For domains that are not of the form above the author shows, by an example, that this assertion need not hold.) The proof makes use of barrier functions and of the fundamental solutions and Green's functions for L and L_0.

L. Nirenberg (NewYork, N.Y.)

1843:

Friedman, Avner. Liouville's theorem for parabolic equations of the second order with constant coefficients. Proc. Amer. Math. Soc. 9 (1958), 272–277.

Consider the parabolic differential equation

$$\frac{\partial u(x, t)}{\partial t} = \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 u(x, t)}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i \frac{\partial u(x, t)}{\partial x_i} + cu(x, t),$$

where the a_{ij}, b_i, and c are real constants and (a_{ij}) is a positive matrix. Let $u(x, t)$ be a solution defined and nonnegative in the half-space $-\infty < t \leq 0$. Here $x = (x_1, x_2, \cdots, x_m)$ and $|x| = (\sum_{i=1}^{m} x_i^2)^{\frac{1}{2}}$. Set $\lim_{t \to -\infty} [\log u(0, t)]t^{-1} = c + \gamma$ (it is shown that this limit necessarily exists). The

author proves that if $c+\gamma \geqq 0$, $\gamma>0$, and if

$$\alpha>(c+\gamma)\left\{\left(\gamma+\frac{b^2}{4}\right)^{1/2}-\frac{b}{2}\right\}^{-1},$$

where $b=(\sum_{i=1}^{n} b_i^2)^{1/2}$, then $u(x, t)$ is unbounded in the cone $|x| \leqq \alpha|t|$, $t \leqq 0$. *I. I. Hirschman, Jr.* (St. Louis, Mo.)

1844:

Nelson, Edward. An existence theorem for second order parabolic equations. Trans. Amer. Math. Soc. **88** (1958), 414–429.

Based upon a probabilistic interpretation of a diffusion equation concerning the continuity property of the sample function of the corresponding Markov process, the author proves the existence theorem: Let $a^{ij}(x)$ and $b^i(x)$ $(i, j=1, \cdots, n)$ be real C^3 functions defined on a connected open set X of the Euclidean n-space such that $(a^{ij}(x))$ is a positive symmetric matrix on X. Then there is, on the Banach space $B(X)$ of the totality of bounded measurable real functions on X, a contraction semi-group P^t $(0 \leqq t<\infty)$ of non-negative linear transformations such that, whenever f is of class C^2 with compact support in X,

$$\lim_{t \downarrow 0} t^{-1}(P^t f(x)-f(x))=\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b^i(x)\frac{\partial f}{\partial x_i}$$

uniformly in X.

The proof of this existence theorem is given firstly for the case where X is an open unit sphere [cf., for the case of compact Riemannian space X, the reviewer's note: Nagoya Math. J. **3** (1951), 1–4; MR **13**, 560], then the general case is treated by piecing together the corresponding local diffusions in each of the coordinate neighbourhoods of X. *K. Yosida* (Tokyo)

1845:

Pogorzelski, W. Propriétés des dérivées tangentielles d'une intégrale de l'équation parabolique. Ricerche Mat. **6** (1957), 162–194.

Let $\Gamma(P, t; Q, \tau)$ be a fundamental solution for the parabolic equation $a_{ij}\partial^2 u/\partial x^i \partial x^j+b_i \partial u/\partial x^i+cu-\partial u/\partial t=0$, where a_{ij}, b_i, c are continuous functions of $(P, t)=(x^1, \cdots, x^n, t)$ on a product set $\bar{X} \times [0, T]$, where $\bar{X}$ is the closure of a domain X in x-space bounded by a closed hypersurface S. It is assumed that a_{ij} is uniformly Hölder continuous with respect to (P, t), that b_i and c are uniformly Hölder continuous with respect to P, that S is of class C^1 and that its unit normal vector is uniformly Hölder continuous on S, finally, that (a_{ij}) is positive definite. Let $\varphi(Q, \tau)$ be continuous on $S \times [0, T]$ and uniformly Hölder continuous with respect to Q. Let $P \in S$ and let the x^1-direction be tangent to S at P. Then the generalized potential

$$U(A, t)=\int_0^t \iint_S \varphi(Q, \tau)\Gamma(A, t; Q, \tau)\, dQ d\tau$$

for $A \in X$ has a partial derivative satisfying: $\lim \partial U(A, t)/\partial x^1$ exists as $A \to P$ and is

$$\int_0^t \iint_S \varphi(Q, \tau)\partial\Gamma(A, t; Q, \tau)/\partial x^1\, dQ d\tau,$$

where the integral is conditionally convergent. The main part of the proof involves an estimate of the contribution of a neighborhood S' of P to the last integral. This estimate is obtained by replacing Γ by its singular part and S' by a piece of the tangent hyperplane. Other results deal with a priori Hölder continuity (with respect to P and/or t) of the tangential derivative.

P. Hartman (Baltimore, Md.)

1846:

Pini, Bruno. Sul problema di Dirichlet per le equazioni a derivate parziali lineari ellittiche in due variabili. Rend. Sem. Mat. Univ. Padova **26** (1956), 177–200.

The author treats the Dirichlet problem for an elliptic equation of order $2m$

(*) $Lu=f$, $(\partial/\partial n)^k u=f_k$ on boundary B, $0 \leqq k \leqq m-1$,

in a bounded domain D (here n represents the interior normal), assuming certain smoothness conditions for the boundary curve and for the coefficients in the differential operator L. For the inhomogeneous biharmonic equation $\Delta^2 u=f$, he sketches, with the aid of a formula due to A. Pleijel [Proc. symposium on spectral theory and differential problems, Oklahoma Agric. and Mech. Coll., 1951, pp. 413–454; MR **13**, 949] a new derivation of the a priori estimates of C. Miranda [Giorn. Mat. Battaglini (4) **2(78)** (1948), 97–118; MR **10**, 706] which assert that

$$|u_x|+|u_y| \leqq \text{const}\,[\max |f|+\max_B (|u_x|+|y_y|)].$$

For an equation of the form $\Delta^2 u+$lower order terms $=f$ he obtains a similar inequality under the assumption of uniqueness of the solution. He shows in addition that, if uniqueness holds for regular (sufficiently differentiable) solutions, then it holds for solutions satisfying the boundary data in some generalized (L_1) sense. Assuming uniqueness, the existence of such solutions is then shown, and also, in case $f_0 \in C^1$, $f_1 \in C^0$, the existence of a solution belonging to C^2 in $D+B$.

In treating the general case (*), for which uniqueness is assumed, the author derives similar results; here, however, certain derivatives of the f_k are required not merely to be continuous but to satisfy a Hölder condition. These results are derived with the aid of the Green's function. {See, more generally, S. Agmon [Comm. Pure Appl. Math. **10** (1957), 179–239].}

L. Nirenberg (New York, N.Y.)

1847:

Mizohata, Sigeru. Le problème de Cauchy pour les équations hyperboliques. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 83–90.

Construction des opérateurs de Leray [Hyperbolic differential equations, Inst. Advanced Study, Princeton, 1953; MR **18**, 313] pour des indices quelconques. Application au problème de Cauchy. (Les opérateurs analogues pour les systèmes p-paraboliques au sens de Petrowsky ont été construits par l'Auteur [J. Math. Soc. Japan **8** (1956), 269–299; MR **19**, 285].) *J. L. Lions* (Nancy)

1848:

Prudnikov, A. P. A solution of a mixed problem in the integral form for a system of two parabolic differential equations. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 869–871. (Russian)

The author studies the system of equations

$$\frac{\partial U}{\partial t}=a\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)+b\frac{\partial V}{\partial t},$$

$$\frac{\partial V}{\partial t}=a'\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+b'\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right),$$

$$(a+a'+bb')^2 \neq 4aa'.$$

By the use of Fourier series and Laplace transformation,

a solution of the initial and boundary value problem

$$U(x, y, 0)=F_1(x, y); \qquad V(x, y, 0)=F_2(x, y)$$
$$U_x'(0, y, t)=U_y'(x, 0, t)=V_x'(0, y, t)=V_y'(x, 0, t)=0$$
$$U_x'(c, y, t)=\psi_1(y, t); \qquad U_y'(x, d, t)=X_1(x, t) \quad (t\geqq 0)$$
$$V_x'(c, y, t)=\psi_2(y, t); \qquad V_y'(x, d, t)=X_2(x, t) \quad (t\geqq 0),$$

is obtained explicitly. The solution admits a simple representation in terms of theta-functions.

R. Finn (Berlin)

1849:

Èĭdel'man, S. D. Cauchy problem for non-linear and quasilinear parabolic systems. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 930–932. (Russian)

The author studies the Cauchy problem (1) $u|_{t=t_0}=\varphi(x)$ for the non-linear system of equations

$$\frac{\partial u}{\partial t}=F\Big(t, x, u, \cdots, \frac{\partial^{2b}u}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}}\Big); \tag{2}$$

here $u=(u_1, \cdots, u_N)$, $x=(x_1, \cdots, x_n)$, and F denotes a column vector. It is assumed throughout that the system (2) is parabolic in the sense of Petrowski in the region $\Pi_1\{t_0\leqq t\leqq T;\ -\infty<x_s<\infty;\ |\partial^m u/\partial x_1^{k_1}\cdots\partial x_n^{k_n}|\leqq M,\ m=0, 1, \cdots, 2b\}$, i.e., that all roots λ of the equation

$$\det\left\{\left\|\sum_{\Sigma k_r=2b}\frac{\partial F_i}{\partial\{\partial^{2b}u_j/\partial x_1^{k_1}\cdots\partial x_n^{k_n}\}}(i\sigma_1)^{k_1}\cdots(i\sigma_n)^{k_n}\right\|_{i,j=1}^{N}-\lambda E\right\}=0$$

satisfy $\mathrm{Re}\,\lambda<-\delta$ for $(x, t, u, \cdots, \partial^{2b}u/\partial x_1^{k_1}\cdots\partial x_n^{k_n})\in\Pi_1$ and all real $\sigma_1, \cdots, \sigma_n$ with $\sum_{s=1}^{n}\sigma_s^2=1$.

Under suitable conditions of smoothness for the vector function $F(t, x, y_1, \cdots, y_l)$ as functions of its arguments, and on the initial data, the author proves the existence of a solution of (1), (2), in an interval $t_0<t\leqq t_0+\Delta$. The solution is shown to be unique in a suitable class of functions, and to depend continuously on the initial data. The complete statement is too complicated to reproduce here, but the result includes, in particular, the following theorem. Let $F(t, x, y_1, \cdots, y_l)$ have $r_1=2b+3$ Lipschitzian derivatives in x, y, which are continuous in t uniformly for all x, y of Π_1; let $\varphi(x)$ have $4b+1$ Hölder continuous derivatives. Then there is a solution $u(x, t)$ with $4b+1$ bounded derivatives in x, which is unique in a somewhat broader class of functions. If $u_1(x, t)$, $u_2(x, t)$ are solutions corresponding to initial data $\varphi_1(x)$, $\varphi_2(x)$, and if $|D_x^s(\varphi_1(x)-\varphi_2(x))|<\varepsilon$, $s=0, 1, \cdots, 2b$, then $|u_1-u_2|<M_1\varepsilon$, where M_1 depends only on the equation and on the space Π_1. {It seems to the reviewer that a bound on the first $2b$ derivatives of $\varphi(x)$ is also necessary in order that the solution should lie in the space Π_1, but the author does not state this.}

In the case that the system (2) is "quasi-linear" or "nearly-linear" in a sense made precise by the author, improved results are obtained. {The reviewer remarks that the statement of Theorem 3 apparently contains a misprint, for the hypotheses as stated are redundant.}

Proofs are based on successive approximations and on estimates for linear systems obtained previously by the author [Mat. Sb. N.S. **38(80)** (1956), 51–92; MR **17**, 857]. Details are not included. *R. Finn* (Berlin)

1850:

Borok, V. M. The reduction of a system of linear partial differential equations with constant coefficients to the normal form. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 13–16. (Russian)

The author studies a system of partial differential equations of the form

$$\frac{\partial^{m_i}u_i(x_1, \cdots, x_n, t)}{\partial t^{m_i}}=\sum_{j=1}^{N}\sum_{\langle m_0, \cdots, m_n\rangle}A_{m_0\cdots m_n}^{ij}\frac{\partial^{m_0+\cdots+m_n}u_j(x_1, \cdots, x_n, t)}{\partial t^{m_0}\partial x_1^{m_1}\cdots\partial x_n^{m_n}} \tag{1}$$
$$(i=1, \cdots, N),$$

where the summation is performed for all indices $\langle m_0, \cdots, m_n\rangle$ with $m_0<n_j$; $m_i<M$ $(i=1, \cdots, n)$. The coefficients $A_{m_0\cdots m_n}^{ij}$ are assumed constant.

Let $\lambda_1(s), \cdots, \lambda_{N_1}(s)$ $(N_1=\sum_{i=1}^{N}n_i;\ s=(s_1, \cdots, s_n))$ be the roots of the characteristic polynomial associated with (1),

$$\det\left(\left\|\begin{matrix}\lambda^{n_1} & & O\\ & \ddots & \\ O & & \lambda^{n_N}\end{matrix}\right\|-\left\|\sum_{\langle m_0, \cdots, m_n\rangle}A_{m_0\cdots m_n}^{ij}\lambda^{m_0}(is_1)^{m_1}\cdots(is_n)^{m_n}\right\|\right). \tag{2}$$

Here the $P_i(s)$ are polynomials in $s_1, \cdots, s_n$. Let $|s|=\sqrt{(|s_1|^2+\cdots+|s_n|^2)}\geqq 1$. The quantity $p_0=\inf k$ for all k such that $\max_{1\leq j\leq N_1}\mathrm{Re}\,\lambda_j(s)\leqq c_k|s|^k$ is called the reduced order of the system (1). The author proves that $p_0=\max_{1\leq i\leq N_1}p_i/i$, where p_i is the highest power in the variables $s_1, \cdots, s_n$ which appears in the polynomial $P_i(s)$ of (2).

An immediate consequence of this theorem is that the reduced order of (1) is always rational, and we may write $p_0=m/k$, m and k relatively prime. A system (1) is said to be of normal type provided i) all $m_0=0$, ii) $n_i\leqq k$, iii) $\sum_{i=1}^{n}m_i\leqq m$. The author proves that the Cauchy problem for any system (1) can be reduced to a Cauchy problem for a system of equations in normal form.

The special cases $p_0=0$ and $p_0=1$ are noted. If $p_0=0$, then (1) can be transformed to a system of ordinary differential equations. If $p_0=1$, then a system of first order is always possible This is the case, in particular, if the system (1) is hyperbolic. *R. Finn* (Berlin)

1851:

Gusačenko, G. M. On the existence of solutions to Cauchy problem for a certain class of partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 27–30. (Russian)

The initial value problem (1) $u(x, 0)=u_0(x)$ for the equation

$$\frac{\partial}{\partial t}P_1\Big(\frac{1}{i}\frac{\partial}{\partial x}\Big)u(x, t)=P_2\Big(\frac{1}{i}\frac{\partial}{\partial x}\Big)u(x, t) \tag{2}$$

is studied by the method of Gel'fand and Šilov [Uspehi Matem. Nauk (N.S.) **8** (1953), no. 6(58), 3–54; MR **15**, 867]. Here P_1 and P_2 are polynomials, presumably with constant coefficients. The author proves the theorem: If the order of $P_1(i^{-1}\partial/\partial x)$ is larger than the order of $P_2(i^{-1}\partial/\partial x)$, and if $u_0(x)$ satisfies the inequality,

$$|u_0(x)|<A_1\exp[-A_2|x|^{q/(q+1)}], \tag{3}$$

where q is the largest multiplicity of the characteristic roots of $P_1(s)$, then there exists a solution of (2) satisfying (1) in the class of functions for which

$$|u(x, t)|\leqq B_1\exp[B_2|x|^{q/(q+1)}].$$

The author states that except for (3), no assumption is made on the smoothness of $u_0(x)$ and that the solution obtained nevertheless satisfies (1) and (2) in a classical sense.

{The precise nature of the result and its significance are not clear to the reviewer. Note that even in the classical case $u_{tx}=0$, the problem (1), (2) is underdetermined. It seems regrettable that the author has not stated precisely what is meant by "classical solution" or by "arbitrary function satisfying (3)". The method seems to require that the initial function be at least measurable, and very likely something more. In the example $u_{tx}=0$, the most general solution of (1), (2) has the form $u(x, t)=u_0(x)+f(t)$ which shows that no differentiable solution can exist unless $u_0(x)$ is at least once differentiable.}

R. Finn (Berlin)

1852:

Čerskiĭ, Yu. I. The reduction of mixed boundary problems to the Riemannian boundary problem. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 927–929. (Russian)

The author studies a boundary value problem for an unknown function $u(x, y)$ in a domain D bounded by segments $y=y_k$, $x_j<x<x_{j+1}$ parallel to the x-axis. On these segments linear relations for the limiting values $u(x, y_k-0)$, $u(x, y_k+0)$, $\partial u(x, y_k-0)/\partial y$, $\partial u(x, y_k+0)/\partial y$ are imposed. In D, $u(x, y)$ is to satisfy a partial differential equation with coefficients independent of x. The problem is solved by the introduction of certain arbitrary functions, which, after Fourier transformation, are determined as the solution of a Riemann-Hilbert problem [cf. Muskhelishvili, "Singular integral equations", Noordhoff, Groningen, 1953; MR **15**, 434]. The author applies his method to the equations $\partial^2u/\partial x^2+\partial^2u/\partial y^2-k^2u=0$ and

$$\sum_{\mu=0}^{p}\sum_{\nu=0}^{q} a_{\mu\nu}(y)\frac{\partial^{\mu+\nu}}{\partial x^{\mu}\partial y^{\nu}}u(x, y)=0.$$

R. Finn (Berlin)

1853:

Mihaĭlov, V. P. The analytical solution of the Goursat problem for a system of differential equations. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 450–453. (Russian)

The author considers the problem of finding the analytic solution $(u_1, \cdots, u_n)$, in some domain, of the system of equations

$$(1)\quad F_i(x, t, u_1, \cdots, u_n, \frac{\partial u_1}{\partial x}, \cdots, \frac{\partial u_n}{\partial x})=0 \qquad (i=1, 2, \cdots, n),$$

such that

$$(2)\quad u_i(x, t)|_{l_i}=0 \quad (i=1, 2, \cdots, n),$$

where l_i is the straight line $x=\mu_i t$. Here, the functions F_i are assumed to be analytic and by expanding F_i in Taylor series in $\partial u_k/\partial x$ and $\partial u_k/\partial t$, the problem leads to the system

$$(3)\quad \frac{\partial u_k}{\partial t}=\sum_{j=1}^{n} b_{kj}\frac{\partial u_j}{\partial x}+\Phi_k \quad (k=1, 2, \cdots, n),$$

where the b_{kj} are real. This in turn leads to a series solution of the problem. The Cauchy-Riemann equations subject to conditions on two lines through the origin are given as an example. *C. G. Maple* (Ames, Iowa)

POTENTIAL THEORY

See also 1845, 2011, 2044.

1854:

Itô, Jun-iti. Asymptotic properties of subharmonic functions of order less than one. Proc. Amer. Math. Soc. **9** (1958), 419–428.

Growth theorems concerning subharmonic functions are established. Central among the results is the following theorem. Let u be subharmonic in the finite plane, let $M(r)=\max_{|z|=r} u(z)$, $m(r)=\inf_{|z|=r} u(z)$, $r>0$. Suppose that $\alpha=\limsup_{r\to\infty} r^{-\rho}M(r)<+\infty$, $0<\rho<1$. Let $\mu^*(r)$ denote $\mu[|z|<r]$, where μ is the mass distribution occurring in the standard "Weierstrassian" representation of u. Suppose that for each $\varepsilon>0$ there exists a positive number N such that

$$\int_{r_1}^{r_2} r^{-1-\rho}\{m(r)\sin\pi\rho-\pi\mu^*(r)\cos\pi\rho\}dr<\varepsilon$$

for $r_2>r_1>N$. Then $\lim_{r\to\infty} r^{-\rho}M(r)$ exists. A number of related results are also obtained.

M. H. Heins (Urbana, Ill.)

1855:

Práger, Milan. Schwarzscher Algorithmus für polyharmonische Funktionen. Apl. Mat. **3** (1958), 106–114. (Czech. Russian and German summaries)

The Schwarz algorithm is a method for solving the Dirichlet problem in the union and in the intersection of two regions whenever it can be solved in each region separately. This paper is concerned with the equation $\Delta^p u=0$ on regions with a piecewise smooth boundary in n-space. The author shows that the Schwarz algorithm leads to a sequence that converges (uniformly with arbitrarily many derivatives) to the solution. The proof uses Hilbert space methods.

J. J. Kohn (Waltham, Mass.)

1856:

Elianu, Jean. Le théorème de Hodge pour les formes harmoniques autoadjointes. Rev. Math. Pures Appl. **2** (1957), 547–549.

The following theorem is proved: In order that there should exist on a compact Riemannian space V, a self-adjoint harmonic form having arbitrarily assigned periods on the linearly independent cycles, it is necessary and sufficient that a) V should be orientable and of dimension $n=4m$; b) the form should be of degree $p=2m$; c) the harmonic components of the cycles should be self-adjoint.

W. V. D. Hodge (Cambridge, England)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 1652, 1733, 1752, 1784, 1785.

1857:

Hosszú, Miklós. Generalization of some functional equations with several variables. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 439–449. (Hungarian)

Zusammenfassung der Resultaten des Verf. in Studia Math. **14** (1953), 100–106 [MR **15**, 962]; Publ. Math. Debrecen **3** (1953–54), 83–86; (1954–55), 205–214 [MR **15**, 962; **17**, 236]; Acta Math. Acad. Sci. Hungar. **4** (1953), 159–167 [MR **15**, 324]; Acta Sci. Math. Szeged **15** (1954), 203–208 [MR **16**, 371]; und Aczél und Hosszú, Acta Math. Acad. Sci. Hungar. **7** (1956), 327–338 [MR **19**, 41].

J. Aczél (Debrecen)

1858:

Schöbe, Waldemar. Gebrochen-iterierte Exponentialfunktion im Reellen. Z. Angew. Math. Mech. **38** (1958), 190–194.

1859:

Popoviciu, Tiberiu. Sur quelques équations fonctionnelles. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Şti. Ser. I. **6** (1955), no. 3–4, 37–49. (Romanian. Russian

and French summaries)

Let $V\begin{pmatrix} f_0, & f_1, & \cdots, & f_n \\ x_1, & x_2, & \cdots, & x_{n+1} \end{pmatrix}$ denote the determinant of values $f_{j-1}(x_i)$, $i, j=1, 2, \cdots, n+1$, and let

$$\Delta_h(x; f_0, f_1, \cdots, f_n)=V\begin{pmatrix} f_0, & f_1, & \cdots, & f_n \\ x, & x+h, & \cdots, & x+nh \end{pmatrix}.$$

In the first part of the present paper, the author shows that if the functions $f_i(x)$, $i=0, 1, \cdots, n-1$, are continuous and satisfy $V\begin{pmatrix} f_0, & f_1, & \cdots, & f_{n-1} \\ x_1, & x_2, & \cdots, & x_n \end{pmatrix} \neq 0$ for each set of n distinct points x_i, $i=1, 2, \cdots, n$, in the interval $[a, b]$, then the general solution of the functional equation $\Delta_h(x; f_0, f_1, \cdots, f_{n-1}, f)=0$, $x, x+nh \in [a, b]$, is of the form $f(x)=c_0f_0(x)+\cdots+c_{n-1}f_{n-1}(x)$, where the c_i are arbitrary constants. [See the following review.]

Let $D\begin{pmatrix} x_1, & \cdots, & x_{m+1} \\ y_1, & \cdots, & y_{m+1} \end{pmatrix}; F\Big)$ denote the determinant of values $F(x_i, y_j)$, $i, j=1, \cdots, m+1$. A function of the form $F(x, y)=\sum_{i=1}^{m} f_i(x)g_i(y)$ is said to be a quasi polynomial of degree m; the degree is said to be effective if the functions $f_i(x)$, and also the functions $g_i(y)$, are linearly independent. In the second part of the paper, it is shown that quasi polynomials of degree m are characterized by the functional equation $D\begin{pmatrix} x_1, & \cdots, & x_{m+1} \\ y_1, & \cdots, & y_{m+1} \end{pmatrix}; F\Big)=0$. Further, it is shown that if $F(x, y)$ is continuous with respect to x and with respect to y (separately) in $a \leqq x \leqq b, c \leqq y \leqq d$, satisfies $D\begin{pmatrix} x_1, & \cdots, & x_m \\ y_1, & \cdots, & y_m \end{pmatrix}; F\Big) \neq 0$ for arbitrary distinct points $x_1, \cdots, x_m$ in $[a, b]$ and for arbitrary distinct points $y_1, \cdots, y_m$ in $[c, d]$, and satisfies

$D\begin{pmatrix} x, & x+h, & \cdots, & x+mh \\ y, & y+k, & \cdots, & y+mk \end{pmatrix}; F\Big)=0$ for arbitrary x, y, h, k such that $x, x+mh \in [a, b]$, $y, y+mk \in [c, d]$, then $F(x, y)$ is a quasi polynomial of degree m and the degree is effective. *E. F. Beckenbach* (Los Angeles, Calif.)

1860:

Rado, Francisc. Conditions de dépendance linéaire pour trois fonctions. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Şti. Ser. I. **6** (1955), no. 3–4, 51–63. (Romanian. Russian and French summaries)

In this paper, the author shows that the functions $F(x)$, $G(x)$, and $H(x)$, defined and continuous for $-\infty < x < +\infty$, are linearly dependent provided that (i) they vanish simultaneously only at isolated points and (ii) they satisfy the equation

$$\begin{vmatrix} F(x) & F(x+h) & F(x+2h) \\ G(x) & G(x+h) & G(x+2h) \\ H(x) & H(x+h) & H(x+2h) \end{vmatrix}=0$$

for all real x and h. [See the preceding review.] The result follows from the theorem that if

$$a_n=\begin{pmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{pmatrix} \quad (n=0, 1, \cdots)$$

is a sequence of vertical vectors, with α_n, β_n, and γ_n real numbers satisfying $\alpha_n^2+\beta_n^2+\gamma_n^2 \neq 0$, and if the determinant $|a_{n-k}\ a_n\ a_{n+k}|$ vanishes for $0<k \leqq n$, $n=1, 2, \cdots$, then in the sequence $\{a_n\}$ there is a subsequence such that (i) the determinant formed by any three arbitrary vectors of the subsequence vanishes and (ii) of any two consecutive terms of the sequence $\{a_n\}$ at least one appears in the subsequence.

E. F. Beckenbach (Los Angeles, Calif.)

1861:

Ghermănescu, M. Equations fonctionnelles linéaires à argument fonctionnel n-périodique. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 43–78. (Romanian. Russian and French summaries)

[Cf. same Bul. **3** (1951), 245–259; MR **15**, 713, where the nonperiodic case was studied.] The author considers operators θ defined on some set in n-space, and their iterates θ_k. If f is a point function he writes $f^k(M)=f[\theta_k(M)]$ (k a positive or negative integer) and calls f functionally continuous. {It seems to the reviewer that all single-valued functions are functionally continuous: the example cited by the author, $\theta(x)=-x$, $f(x)=\log x$, succeeds in failing to have $f^2(x)=f(x)$, as it should, only if one chooses different branches of the logarithm at different times.} The operator θ is n-periodic when $\theta_n=\theta$; only such operators are considered in this paper. First the author considers homogeneous linear equations with constant coefficients, $E(f)=a_0f+a_1f^1+\cdots+a_{n-1}f^{n-1}=0$. The characteristic equation is $E(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}=0$; $E(f)=0$ has a nontrivial solution only if the characteristic equation has an nth root of unity as one of its roots. In this case let $D(x)$ be the highest common factor of $1-x^n$ and $E(x)$, and $F(x)=(1-x^n)/D(x)$; then $f=F(u)$ with arbitrary u is the general solution of $E(f)=0$. Numerous special cases are discussed, for example $f(M)=\alpha f^k(M)$. Next the author considers nonhomogeneous equations $E(f)=g(M)$. This is soluble if $E(x)$ is prime to $1-x^n$; otherwise, if and only if $F(g)=0$ with F as above. Example: $f-3f^1+2f^2=g$ is soluble only if $g+g^1+g^2=0$, and the solutions are all of the form

$$f(M)=\frac{1}{21}[4g(M)+5g^1(M)]+u(M)+u^1[M]+u^2[M],$$

where u is arbitrary. After considering analogous results for multiplicative instead of additive equations, and further examples, the author turns to "functionally discontinuous" solutions. {These seem to be formal solutions which exist only because the author insists on having $\log(-x)=\log x+i\pi$ even when x is negative; thus (to take a simple special case) he claims that $f(-x)-f(x)=i\pi$ has solutions $f(x)=u(x)+\log v(x)$ with u even and v odd.} The author stresses a representation that generalizes the familiar representation of any function as the sum of even and odd parts: if α is an nth root of unity and $g_k(M)=n^{-1}\{g(M)+\alpha^k g^1(M)+\cdots+\alpha^{(n-1)k}g^{n-1}(M)\}$, so that $g_k(M)=\alpha^{-k}g_k^1(M)$, then $g(M)=g_0(M)+g_1(M)+\cdots+g_{n-1}(M)$. *R. P. Boas, Jr.* (Evanston, Ill.)

1862:

Ghermănescu, M. Fonctions doublement automorphes. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 253–260. (Romanian. Russian and French summaries)

The functions of the title have domains in n-space (points M) and satisfy two equations $f[\theta_1(M)]=\lambda_1(M)f(M)$, $f[\theta_2(M)]=\lambda_2(M)f(M)$, where θ_1 and θ_2 commute and λ_1, λ_2 are given functions, which must satisfy

$$(*) \qquad \lambda_1[\theta_2(M)]\lambda_2(M)=\lambda_2[\theta_1(M)]\lambda_1(M);$$

this condition is shown to be sufficient and a formal

representation is given for f. The solutions λ_1, λ_2 of (*) have the form $\lambda_1(M)=\phi[\theta_1(M)]\phi_2(M)/\phi(M)$ (for λ_2 interchange 1 and 2), where ϕ is arbitrary and ϕ_f is invariant under θ_f. *R. P. Boas, Jr.* (Evanston, Ill.)

1863:

Ghermănescu, M. Equations fonctionnelles de matrices. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **9** (1957), 261–285. (Romanian. Russian and French summaries)

[Cf. Com. Acad. R. P. Romîne **5** (1955), 489–498; MR **17**, 44.] With notations as in the preceding two reviews, the author considers systems of functional equations, which he treats as matrix equations, principally (1) $\Phi(M_1)=A(M)\Phi(M)$, where $A(M)$ is a given matrix, Φ is unknown, and $M_1=\theta(M)$, $M_n=\theta(M_{n-1})$; (2) $\Phi(M_1)=A(M)\Phi(M)+B(M)$. A particular solution of (1), from which all solutions can be obtained, is the infinite product $G(M)=A^{-1}(M)A^{-1}(M_1)\cdots$, if this converges uniformly. For (2), let $\Phi_1(M)$ be any solution of (1); then $\Phi_1(M)C(M)$ satisfies (2) if $C(M)$ satisfies $C(M_1)=C(M)+D(M)$, where $\Phi_1(M_1)D(M)=B(M)$; $C(M)$ is obtainable from $-C(M)=\sum_0^\infty D(M_n)$ if this converges; if not, it can be made to converge by adding appropriate constant matrices to its terms. There are similar results for the nth order equations

$$\Phi(M_n)+A_1(M)\Phi(M_{n-1})+\cdots+A_n(M)\Phi(M)=0 \text{ or } B(M).$$

R. P. Boas, Jr. (Evanston, Ill.)

SEQUENCES, SERIES, SUMMABILITY

1864:

***Green, J. A. Sequences and series.** Library of Mathematics. The Free Press, Glencoe, Ill; Routledge and Kegan Paul Ltd., London; 1958. viii+78 pp. $1.25; 5s.

"This book is intended primarily for students of science and engineering. . . More emphasis is laid on the illustration of basic ideas by numerical examples, than on formal proofs." *From the preface*

1865:

Carlitz, Leonard. A note on the Rogers-Ramanujan identities. Math. Nachr. **17** (1958), 23–26.

The Rogers-Ramanujan identities [for proof and references, see Hardy, "Ramanujan", Macmillan, New York, 1940; MR 3, 71; chap. 6]:

$$\sum_0^\infty \frac{q^{m^2}}{(q)_m}=\prod_0^\infty (1-q^{5m+1})^{-1}(1-q^{5m+4})^{-1},$$

$$\sum_0^\infty \frac{q^{m^2+m}}{(q)_m}=\prod_0^\infty (1-q^{5m+2})^{-1}(1-q^{5m+3})^{-1},$$

where $(q)_m=(1-q)(1-q^2)\cdots(1-q^m)$, $(q)_0=1$, are equivalent to

$$\sum_0^\infty \frac{q^{m^2}}{(q)_m}=\sum_{-\infty}^\infty (-1)^m q^{\frac{1}{2}(5m^2+m)}\prod_1^\infty (1-q^n)^{-1},$$

$$\sum_0^\infty \frac{q^{m^2+m}}{(q)_m}=\sum_{-\infty}^\infty (-1)^m q^{\frac{1}{2}(5m^2+3m)}\prod_1^\infty (1-q^n)^{-1},$$

respectively. The author's proof of these identies is a variant of Roger's proof as given by Hardy. His method depends upon the basic analog of the Bessel function first defined by Jackson [Proc. London Math. Soc. (2) **2** (1905), 192–220] and later discussed by Hahn [Math. Nachr. **2** (1949), 340–379; MR **11**, 720]. However, no properties of Jackson's function are assumed. The advantage of the author's procedure is that it eliminates some of the artificial features of previous proofs.

A. L. Whiteman (Los Angeles, Calif.)

1866:

Pati, T. Products of summability methods and Mercerian transformations. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 514–521.

Ist (H, χ) ein reguläres Hausdorffverfahren mit $U_\alpha=\alpha E+(1-\alpha)(H, \chi)\approx E$, so gilt $AU_\alpha\approx A$ für die Verfahren $A(x, s_n)=\sum p_n s_n x^n/\sum p_n x^n$ in den Fällen

$$p_n=(-1)^n\binom{-k}{n},\ k>0\ (x\to 1-0);\ p_n=1/n!\,(x\to\infty).$$

Entsprechendes gilt für Hausdorffverfahren für Funktionen und Wienerverfahren A.

A. Peyreimhoff (Giessen)

1867:

Aljančić, Slobodan. Classe de saturation des procédés de sommation de Hölder et de Riesz. C. R. Acad. Sci. Paris **246** (1958), 2567–2569.

The author refers to a paper by M. Favard to be published in J. Math. Pures Appl., in which is shown that the phenomenon of saturation of the process of Cesàro in the case of Fourier series has its origin in a theorem of equivalence pertaining to the summation of numerical series. The author of the present paper gives analogous theorems for the Hölder and Riesz processes. Theorems are given without proofs; the following is one such theorem: Let S be the series $\sum u_n$ and T be the series $\sum nu_n$; let $C_n^{(k)}(S)$ and $C_n^{(k)}(T)$ be the nth elements in the sequences of the kth order Cesàro process applied to the series S and T; then for $-1<\alpha<k$,

$$C_n^{(k)}(S)-\sigma=O\left(\frac{\log^\beta n}{n^{\alpha+1}}\right)\Leftrightarrow C_n^{(k)}(T)=O\left(\frac{\log^\beta n}{n^\alpha}\right),$$

where σ is the generalized sum of S by the Cesàro-k process. *D. Moskovitz* (Pittsburgh, Pa.)

1868:

Sokolin, A. S. On certain classes of methods of summation of divergent series. Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1(79), 193–200. (Russian)

Identities involving binomial coefficients are used to obtain relations among several regular methods for evaluation of series. The latter methods include those of Euler, Hardy-Littlewood, Meyer-König, Borel, Vallée-Poussin, and Obreschkoff, and nine others of which two typical ones are the following. When q is a positive integer, a given series $a_0+a_1+\cdots$ is evaluable $M_q^{(1)}$ to S if

$$\lim_{n\to\infty}\sum_{k=0}^n\left[\binom{qn+n-k}{n}\Big/\binom{qn+n}{n}\right]\binom{q+1}{q}^k a_k=S$$

and is evaluable $M_q^{(4)}$ to S if

$$\lim_{n\to\infty}\sum_{k=0}^n\left[\binom{qn+n}{n+k}\Big/\binom{qn+n}{n}\right]q^{-k}a_k=S.$$

A typical result is an identity which shows that the $M_q^{(4)}$ transform of a series is the $M_q^{(1)}$ transform of the Euler transform of order q of the series.

R. P. Agnew (Ithaca, N.Y.)

1869:

Hirokawa, Hiroshi. Further remarks on the paper "Uniform convergence of some trigonometrical series". Tôhoku Math. J. (2) **9** (1957), 110–112.

The author answers a question which he had posed in an earlier paper [same J. **6** (1954), 162–173; MR **17**, 845]. He shows that if $\beta \geqq 1$, there exist a series $\sum_{n=1}^{\infty} a_n$ such that the nth (C, β) sum of $\{na_n\}$ is $o(n/\log n)$, without the series being summable $(R, 1)$. *P. Civin* (Eugene, Ore.)

1870:

Yano, Kenji. Notes on Tauberian theorems for Riemann summability. Tôhoku Math. J. (2) **10** (1958), 19–31.

Die Reihe $\sum_1^{\infty} a_n$ mit den Teilsummen s_n heißt R_1-summierbar [$(R, 1)$-summierbar] zum Wert s, wenn $\lim_{t\to 0+} 2\pi^{-1}\sum_1^{\infty} s_n(\sin nt)/n = s$ [$\lim_{t\to 0+} \sum_1^{\infty} a_n(\sin nt)/nt = s$] ist; beide Verfahren sind nicht permanent. Der Verf. beweist einige Sätze Tauberscher Art, in denen die Riemann-Summierbarkeit von $\sum a_n$ aus Annahmen über

$$S_n^k = \sum_{\nu=0}^{n} \binom{n-\nu+k}{n-\nu} a_\nu \quad (k \text{ reell}, a_\nu \text{ reell})$$

erschlossen wird.

R_1-$\sum a_n = 0$ und $(R, 1)$-$\sum a_n = 0$ gelten immer dann, wenn entweder (i) $\sum_{\nu=n+1}^{2n} |S_\nu^r| = o(n^{1+r\alpha})$ und $\sum_{\nu=n+1}^{2n} (S_\nu^{-s} - S_\nu^{-s}) = O(n^{1-s\alpha})$ $(r>0, s>0, 0<\alpha\leqq 1)$ oder (ii) $\sum_{\nu=n+1}^{2n} |s_\nu| = o(n/\log n)$ und $\sum_{\nu=n+1}^{2n} (|S_\nu^{-s}| - S_\nu^{-s}) = O(n^{1-\delta})$ $(s>0, 0<\delta<1)$ erfüllt sind. Darin sind frühere Ergebnisse von Sunouchi [dasselbe J. **5** (1953), 34–42, 189; MR **15**, 304, 787], Hirokawa und Sunouchi [ibid. **5** (1954), 261–267; MR **16**, 240], und O. Szász [Amer. J. Math. **73** (1951), 779–791; MR **13**, 456] enthalten. *D. Gaier* (Stuttgart)

1871:

Rajagopal, C. T. On the Riemann-Cesàro summability of series and integrals. Tôhoku Math. J. (2) **9** (1957), 247–263; errata, **10** (1958), 366.

O. Szász [Amer. J. Math. **73** (1951), 779–791; MR **13**, 456] a demontré la relation suivante parmi les procédés de sommations d'Abel et de Riemann: $\sum_1^{\infty} a_k = l(A)$ et la condition du type tauberienne $\sum_1^n k|a_k| = O(n) \Rightarrow \sum_1^{\infty} a_k = l(R, 1)$, ou bien $\sum_1^{\infty} a_k = l(R_1)$.

L'auteur considère le problème analogue pour le procédé (R, p, α) introduit par H. Hirokawa [Tôkohu Math. J. **7** (1955), 279–295; MR **17**, 1076]. Ce procédé se réduit au $(R, 1)$ pour $p=1$, $\alpha=-1$ et au (R_1) pour $p=1$, $\alpha=0$. Alors, l'auteur démontre: (I) De $\alpha+1\geqq 0$, (*) $\sum_1^{\infty} a_k = l(C, \alpha+2)$, $\sum_1^n |\sigma_k^\alpha| = O(n)$, $p>\alpha+1$ (p un entier positif, σ_n^α la moyenne arithmétique d'ordre α), résulte $\sum_1^{\infty} a_k = l(R, p, \alpha)$. (I') De plus, dans (I) l'hypothèse (*) peut être remplacé par (**) $\sum a_k = l(A)$, sans aucun autre changement. Il déduit aussi les formes intégrales de (I) et (I'). *M. Tomić* (Beograd)

APPROXIMATIONS AND EXPANSIONS

1872:

***Achieser, N. I. Theory of approximation.** Translated by Charles J. Hyman. Frederick Ungar Publishing Co., New York, 1956. x+307 pp. $8.50.

An English translation of the noted Russian work. The original [OGIZ, Moscow-Leningrad, 1947] was reviewed in MR **10**, 33 and the German translation [Akademie-Verlag, Berlin, 1953] mentioned in MR **15**, 867. An earlier, briefer, version of the original [Kharkoff, 1940] was reviewed in MR **3**, 234.

1873:

Stancu, D. D. Contribution à la dérivation partielle numérique des fonctions de deux ou plusieurs variables. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **8** (1956), 733–763. (Romanian. Russian and French summaries)

While there is a large literature on the numerical differentiation of functions of a single variable, the corresponding problem for several variables has received less attention. The author establishes such formulae, giving approximate values of the partial derivatives, by linear combinations of values of the function and its successive, lower order, partial derivatives at a finite set of points. The corresponding error is a functional $R_s[f]$. A formula is said to have "partial degree of exactness $(m_1, m_2, \cdots, m_s)$" if $R_s[P]=0$ for any polynomial P of s variables and corresponding degrees not in excess of $(m_1, m_2, \cdots, m_s)$, while there exists a polynomial P_1 of degrees $(m_1+1, m_2+1, \cdots, m_s+1)$, for which $R[P_1]\neq 0$. The author gives the degrees of exactness for his formulae of numerical partial differentiation. Newton's, alternatively also Lagrange's, interpolation polynomials are taken as starting points in the proofs. These are elementary in character, but rather complicated. In some of them, only the cases of 2 or 3 variables are worked out, in order to keep notations reasonably simple. In the case of lattice points with unequal distances, use is made of divided differences. Somewhat simpler results are found in the case of equidistant lattice points, but they are still too complicated to be quoted here. As particular cases one finds some known formulae [Schulz, Formelsammlung, Zur praktischen Mathematik, Berlin 1937; Panov, Spravočnik po čislennomu rešeniyu differencial'nyh uravnenii, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR **14**, 93], but the expressions for the error term seem new even in these cases. *E. Grosswald* (Philadelphia, Pa.)

1874:

Stancu, D. D. Généralisation de certaines formules d'interpolation pour les fonctions de plusieurs variables; quelques considérations sur la formule d'intégration numérique de Gauss. Acad. R. P. Romîne. Bul. Ştiinţ. Secţ. Ştiinţ. Mat. Fiz. **9** (1957), 287–313. (Romanian. Russian and French summaries)

The first part contains a generalization of Lagrange's interpolation polynomial in several variables, for rather general lattice points. The remainder is expressed with the help of divided differences. One obtains the classical Lagrange interpolation polynomial as a particular case. As application, the author generalizes to several variables a mean value theorem due to T. Popoviciu [Bull. Math. Soc. Roumaine Sci. **43** (1941), 85–141; MR **7**, 116]. In the second part, the author considers the Lagrange-Hermite interpolation polynomial of degree $2n-1$, which leads to the quadrature formula

$$\int_{-1}^{+1} f(x)dx = \sum_{i=1}^{n} A_i f(x_i) + \sum_{i=1}^{n} B_i f'(x_i) + \rho(f),$$

where $\rho(f)=0$ if f is a polynomial of degree less than $2n$. Proofs are given for the classical results that if for the x_i one takes the roots of $P_n(x)$, the nth Legendre polynomial, then $B_i=0$, and one obtains Gauss' quadrature formula. Similarly, if one uses the ordinary Lagrange interpolation polynomial of degree $n+k$, at points $\{x_i, \lambda_j\}$, $1\leqq i\leqq n$,

$1\leq j\leq k\leq n$, one obtains a quadrature formula $\int_{-1}^{+1} f\,dx = \sum_{i=1}^{n} A_i(fx_i)+\sum_{j=1}^{k} B_j f(\lambda_j)+\rho(f)$ and $B_j=0$, provided that the x_i are selected as roots of $P_n(x)=0$. This result holds also if the λ's are not distinct, or coincide possibly with roots of $P_n(x)$. Analogous results are shown to hold also in the case of quadrature formulae for double integrals over squares; the lattice points have to be selected as (x_i, y_j), $1\leq i\leq n$, $1\leq j\leq m$, where the x_i's are the zeros of $P_n(x)$ and the y_j's are the zeros of $P_m(y)$. Expressions for the coefficients and remainder terms of these formulae are obtained, using divided differences. Further generalizations to several variables are indicated and some particular quadrature formulae are worked out in detail. A generalization of these results to other domains of integration (circle, regular polygon, sphere, regular polyhedron) is promised. *E. Grosswald* (Philadelphia, Pa.)

1875:

Stancu, D. D. La généralisation de la formule de quadrature de Gauss-Christoffel. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **8** (1957), no. 1, 1–18. (Romanian. Russian and French summaries)

Results of preceding paper [#1874 above] are generalized as follows. Let a_i $(1\leq i\leq s)$ be real numbers, to each being attached a multiplicity r_i, $\sum_{i=1}^{s} r_i=p$. Generalizing Gauss' quadrature formula (which corresponds to the case $r_i=p=0$), Christoffel [J. Reine Angew. Math. **55** (1858), 61–82] showed for the case $r_i=1$, $p=s$, that it is possible to select n fixed abscissae x_i in such a way, that the quadrature formula

$$\int_{-1}^{+1} f(x)dx=\sum_{i=1}^{n} A_i f(x_i)-\sum_{i=1}^{s} C_i f(a_i)$$

should be exact, if f is a polynomial of degree $\leq 2n+s-1$. For arbitrary integrable $f(x)$ let $\rho[f]$ be the difference between the two members of the quadrature formula; then the linear functional ρ is said to be of degree of exactness k, if $\rho[f]=0$, whenever f is a polynomial of degree $\leq k$, while $\rho[f]\neq 0$, for at least one polynomial of degree $k+1$. In this terminology, Christoffel's formula with $n+s$ terms is of degree of exactness $2n+s-1$. The author generalizes Christoffel's result for the case of arbitrary r_i's. The corresponding quadrature formulae then involve, beside the values of $f(x)$, also those of its derivatives of orders $\leq r_i-1$, at the points a_i,

$$\int_{-1}^{+1} f dx=\sum_{\nu=1}^{n} A_\nu f(x_\nu)+\sum_{i=1}^{s}\sum_{j=0}^{r_i-1} C_{ij} f^{(j)}(a_i)+\rho[f].$$

If the x_ν are the (distinct) roots of a certain polynomial $U_n(x)$ of degree n (that generalizes Legendre's polynomial), then $\rho[f]$ is shown to be of degree of exactness $2n+p-1$. The proof uses a result by Szegö [Math. Ann. **82** (1921), 188–212]. The author gives several expressions (one of which is an explicit formula) for the A_ν and one for the C_{ij}. Using divided differences, he obtains also a representation for the error term $\rho[f]$. This is new even in the case of Christoffel's general formula [for Gauss' formula, the result is due to A. Markoff, ibid. **25** (1885), 427–432]. Many particular formulae are worked out in detail.

E. Grosswald (Philadelphia, Pa.)

1876:

Gupta, D. P. On order of sum of the series of ultraspherical functions. Tôhoku Math. J. (2) **10** (1958), 42–48.

In 1923 B. M. Wilson proved that the partial sum $S_n(x)$ of the Legendre expansion of a Lebesgue integrable $f(x)$ in $(-1, 1)$ is $o(n^k)$ for $0<k\leq\frac{1}{2}$ and $o(\log n)$ for $k=0$, provided the number k is such that the behavior of $f(x)$ at $x=\pm 1$ insures the existence of the integral of the product $(1-x^2)^{\frac{1}{2}k-\frac{1}{4}}\cdot f(x)$ in $(-1, 1)$.

The author extended this theorem for almost all x in the open interval $(-1, 1)$ to the ultraspherical expansions, the corresponding assumption being the existence of the integral of $(1-x^2)^{\frac{1}{2}k-\frac{1}{2}+\frac{1}{2}\lambda}\cdot f(x)$ in $(-1, 1)$, where λ is the parameter of the family of ultraspherical polynomials $P_n^{(\lambda)}(x)$ defined by their generating function $(1-2xz+z^2)^{-\lambda}$. {This extension is formulated for $0\leq k\leq\lambda\leq\frac{3}{4}$ only ($\lambda=\frac{1}{2}$ corresponds to Wilson's theorem), though the elegant proof seems to hold also for $\frac{3}{4}<\lambda<1$. Wilson's theorem holds for all x in the closed interval $(-1, 1)$, and the restriction to almost all x in the open interval is due to the method of proof and could be avoided. In the opinion of the reviewer the extension should hold for all positive λ.} *E. Kogbetliantz* (New York, N.Y.)

1877:

Alexits, Georges. Sur la convergence absolue de certains développements orthogonaux. Acta Math. Acad. Sci. Hungar. **8** (1957), 303–310.

Aus einer Funktionenfolge $\{\varphi_n(x)\}$, $a\leq x\leq b$, mit $|\varphi_n(x)|\leq K$, $\int_a^b \varphi_n^2(x)dx=1$ wird eine neue Folge $\{\psi_n(x)\}$ für $n\geq 1$ gebildet durch $\psi_n(x)=\varphi_{\nu_1+1}(x)\cdots\varphi_{\nu_m+1}(x)$, $n=2^{\nu_1}+2^{\nu_2}+\cdots+2^{\nu_m}$ $(0\leq\nu_1<\nu_2<\cdots<\nu_m)$. Ferner sei $\psi_0(x)=1$. Wegen $\psi_{2^{n-1}}(x)=\varphi_n(x)$ enthält $\{\psi_n(x)\}$ alle Funktionen $\varphi_n(x)$. Der Verf. bezeichnet $\{\varphi_n(x)\}$ als multiplikativ orthogonal, falls $\int_a^b \varphi_n(x)dx=0$ ist für $n=1, 2, \cdots$. Es gilt nun folgender Satz: Ist $\{\varphi_n(x)\}$ multiplikativ orthogonal und ist $f(x)$ beschränkt und so beschaffen, dass $\int_a^b f(x)\varphi_n(x)dx=0$ ist für $n\neq 2^{k-1}$, so ist

$$\sum\left|\int_a^b f(x)\varphi_n(x)dx\right|=\sum\left|\int_a^b f(x)\psi_{2^{n-1}}(x)dx\right|<\infty.$$

Aus diesem Satz ergibt sich u.a. als Spezialfall eine Verschärfung des bekannten Sidonschen Satzes über lakunäre Fourierreihen. Es werden verschiedene weitere Spezialfälle behandelt. *A. Peyerimhoff* (Giessen)

1878:

Chako, Nicholas. Développement asymptotique d'intégrales doubles que l'on rencontre dans la théorie de la diffraction. C. R. Acad. Sci. Paris **247** (1958), 436–438.

The asymptotic behavior of $\iint g(x, y)\exp(ik\phi(x, y))\,dxdy$ is given for large real k. Although restrictions on the functions in the integrand are somewhat milder, the study of critical points, the results, and the use of van der Corput's neutralizers are essentially those given in a paper by J. Focke [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **101** (1954), no. 3; MR **16**, 919] to which the author refers. *T. E. Hull* (Pasadena, Calif.)

1879:

Riney, T. D. A finite recursion formula for the coefficients in asymptotic expansions. Trans. Amer. Math. Soc. **88** (1958), 214–226.

The asymptotic behavior of $\sum g(n)z^n$ for large z is determined by the coefficients appearing in the asymptotic factorial expansion of $g(w)$ when

$$g(w)=\frac{\prod_{i=1}^{q}\Gamma(w+\sigma_i)}{\prod_{j=0}^{p}\Gamma(w+\rho_j)},$$

where $p\leq q$, and σ_i, ρ_j are arbitrary complex parameters. In an earlier paper [Proc. Amer. Math. Soc., **7** (1956), 245–249; MR **17**, 962] the author used a difference equation satisfied by $g(w)$ to obtain a recurrence

formula for these coefficients. He now obtains a simpler recurrence formula, of fixed length q, by first introducing $\phi(t)$, defined as an inverse transform of $g(w)$, and showing that $\phi(t)$ satisfies a differential equation of the Fuchsian type. Then $\phi(t)$ is expanded about one of the regular singular points of the equation, and the required formula is obtained. Examples are also given.

T. E. Hull (Pasadena, Calif.)

FOURIER ANALYSIS

1880:

Efimov, A. V. On the Fourier coefficients of functions of class $\bar{H}_2^1$. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 303–311. (Russian)

Let $\bar{H}_p^{\alpha}(K)$ be the class of all continuous functions $f(x)$ of period 2π such that

$$|\Delta_t^p f(x)| = \left|\sum_{i=0}^{p} (-1)^{p-i}\binom{p}{i} f(x+it)\right| \leqq Kt^{\alpha},$$

$x \in [0, 2\pi]$, $t>0$. $\bar{H}_1^1(1)$ is the class $\mathrm{Lip}_1 1$ and $\bar{H}_2^1(2)$ is the Zygmund class Λ^*. Let $\pi a_n(f) = \int_0^{2\pi} f(x)\cos nx\,dx$ $(n=1, 2, \cdots)$, and let $\sup_{f\in \bar{H}_2^1(2)} |a_n(f)| = 4c_0/\pi n$. The author proves that $1{,}1335 < c_0 < 1{,}255$. (The corresponding result for the class $\bar{H}_1^{\alpha}(1) = \mathrm{Lip}_1 \alpha$ is due to Lebesgue.) The proof is based on the following lemma: If $f \in \bar{H}_2^1(2)$ and $a_n(f) = A$, then there exists a function $\varphi \in \bar{H}_2^1(2)$ such that $a_n(\varphi) = A$, $\varphi(k\pi/n+x) = \varphi(k\pi/n-x)$, $\varphi((2m+1)\pi/2n+x) = \varphi((2m+1)\pi/2n-x)$, $\varphi((2m+1)\pi/2n) = 0$; $k, m = -n, \cdots, 0, \cdots, n$.

M. Cotlar (St. Louis, Mo.)

1881:

Vigil y Vázquez, Luis. Results on the convergence in measure of trigonometric series. Rev. Acad. Ci. Madrid **51** (1957), 161–163. (Spanish)

Der Verf. gibt (ohne Beweise) eine Reihe von Ergebnissen über die Masskonvergenz von trigonometrischen Reihen und Fourierreihen. Er zeigt u.a. dass eine trigonometrische Reihe $\frac{1}{2}a_0 + \sum_1^{\infty} A_n(x)$ $(A_n(x) = a_n \cos nx + b_n \sin nx,\ a_n \to 0,\ b_n \to 0)$ genau dann Fourierreihe ist, wenn gilt (i) die Reihe ist masskonvergent gegen ein $f(x) \in L(0, 2\pi)$, (ii) die Reihen $\sum_n A_{pn}(x)$, $\sum_n A_{pn+h}(x) + A_{pn-h}(x)$, $h < [p/2]$ sind masskonvergent gegen Funktionen $\varphi_p(x)$, $\varphi_{h,p}(x)$ und diese sind masskonvergent gegen 0 für $p \to \infty$. Dabei folgt die Masskonvergenz der Reihen in (ii) bereits aus (i).

A. Peyerimhoff (Giessen)

1882:

Kennedy, P. B. Remark on a theorem of Zygmund. J. London Math. Soc. **33** (1958), 71–72.

Zygmund has shown that if (1) $n_{k+1}/n_k > \lambda > 1$, and (2) $\alpha_k, \beta_k \to 0$, then the series (3) $\sum \alpha_k \cos n_k x + \beta_k \sin n_k x$ converges on a set which is everywhere dense and everywhere of the power of the continuum [Fund. Math. **16** (1930), 90–107]. The author shows that (1) is best possible in the sense that it cannot be replaced by $n_{k+1}/n_k > 1+\phi(n_k)$, where $\phi(t) \searrow 0$ as $t \to \infty$. In fact, he shows that under this condition there exists an everywhere divergent series of form (3) for which (2) holds. The method is an elaboration of that used by Steinhaus to show that there exist everywhere divergent trigonometric series whose coefficients tend to zero. [J. London Math. Soc. **4** (1929), 86–88].

D. Waterman (Lafayette, Ind.)

1883:

Ul'yanov, P. L. Permutations of a trigonometric system. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 568–571. (Russian)

J. Marcinkiewicz [Studia Math. **6** (1936), 39–45] has shown that if $1 \leqq p < 6/5$, then there is an $f \in L^p[0, 2\pi]$ and a rearrangement of the Fourier series of f such that no sequence of its partial sums is convergent on any subset of $[0, 2\pi]$ of positive measure. The author of the paper under review generalizes this in the following way: If $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is a real trigonometric series for which $a_n^2 + b_n^2 = O(1)$ and if there is an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers for which $\sum_1^{\infty} (a_{n_k}^2 + b_{n_k}^2) = \infty$ and the equations $\pm n_i \pm n_j = n$ have at most A solutions n_i, n_j for each positive integer n, then $\sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)$ is summable by no K-method of summability (a regular Toeplitz method with matrix (α_{ij}) for which $\sum_{j=0}^{\infty} j|\alpha_{ij}| < \infty$ for $i = 0, 1, 2, \cdots$) on any subset of $[0, 2\pi]$ of positive measure, nor is any sequence of partial sums of this series convergent on any subset of $[0, 2\pi]$ of positive measure,

He states further that this result can be used to exhibit $f \in L[0, 2\pi]$ for a given ε, $0 < \varepsilon < \pi$, such that $f(x) = 0$ in $[\varepsilon, 2\pi - \varepsilon]$, $f \in C^{\infty}[\delta, 2\pi - \delta]$ for every δ, $0 < \delta < \pi$, and a rearrangement of the Fourier series of f has the properties asserted of $\sum (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)$ above, i.e., rearranged Fourier series of locally arbitrarily smooth functions can be quite pathological.

A. E. Livingston (Seattle, Wash.)

1884:

Chow, Hung Ching. On the strong summability of Fourier series. J. London Math. Soc. **33** (1958), 161–170.

Let $f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx$ and $s_n = a_1 + a_2 + a_3 + \cdots + a_n$. Let $f_{\alpha}(x)$ $(\alpha > 0)$ be the αth mean of f, i.e. $f_{\alpha}(x) = \alpha x^{-\alpha} \int_0^x (x-t)^{\alpha-1} f(t)\,dt$. The author proves the following theorem: (1) If $\int_0^x |f_{\alpha}(t) - s|^p dt = O(x)$ as $x \to 0$ for $0 < \alpha < \frac{1}{2}$ and $p > 1/(1-\alpha)$ and $f_k(x) \to s$ for some positive k, then $\sum_{n=1}^{m} |s_n - s|^q = o(m)$ for $0 < q < 1/\alpha$. (2) If $\int_0^x |f_{\alpha}(t) - s|^p dt = o(x)$ as $x \to 0$ for $0 < \alpha \leqq \frac{1}{2}$ and $p = 1/(1-\alpha)$ and $f_{\beta}(x) \to s$ for some positive $\beta < 1$, then

$$\sum_{n=1}^{m} n^{-1} |s_n - s|^p = o(\log m).$$

The first part is a generalization of the Hardy-Littlewood theorem [Proc. London Math. Soc. (2) **26** (1927), 273–286] except for the restriction of q.

S. Izumi (New Haven, Conn.)

1885:

Matsumoto, Kishi. A sufficient condition for the absolute Riesz summability of a Fourier series. Tôhoku Math. J. (2) **9** (1957), 222–233.

Let $f(t)$ be an integrable function and $\varphi_{\alpha}(t)$ be the αth Cesàro mean of $\varphi(t) = \frac{1}{2}(f(x+t) + f(x-t))$, i.e., $\varphi_{\alpha}(t) = \alpha t^{-\alpha} \int_0^t (t-u)^{\alpha-1} \varphi(u)\,du$. The author proves that if the function $\varphi_{\alpha}(t)(\log k/t)^{\alpha(\Delta-1)}$ is of bounded variation for $k > \pi e^{\alpha(\Delta-1)+1}$, $\alpha > 0$ and $\Delta \geqq 1$, then the Fourier series of f is summable $|R, \exp(\log w)^{\Delta}, \beta|$ where $\beta > \alpha$. This is a generalization of theorems of Mohanty-Misra [same J. **6** (1954), 5–12; MR **16**, 465], Kinukawa [unpublished] and Pati [Trans. Amer. Math. Soc. **76** (1954), 351–374; MR **15**, 952]. The case $\Delta = 1$ reduces to the Bosanquet theorem concerning absolute Cesàro summability [Proc. London Math. Soc. **41** (1936), 517–528]. Proof is given for $\Delta > 1$

and in this case it is equivalent to prove that

$$\int_2^\infty \frac{(\log w)^{\Delta-1}}{we^{\beta(\log w)^\Delta}} \times \Big| \sum_{n<w} (e^{(\log w)^\Delta} - e^{(\log n)^\Delta})^{\beta-1} e^{(\log n)^\Delta} A_n(x) \Big| dw < \infty,$$

where $A_n(x)$ is the nth term of the Fourier series of f.

S. Izumi (New Haven, Conn.)

1886:

Walmsley, C. On generalized Fourier series. Rend. Circ. Mat. Palermo (2) **6** (1957), 141–156.

Der Verf. betrachtet verallgemeinerte Fourierreihen $\frac{1}{2}a_0+\sum_1^\infty (a_n \cos nx+b_n \sin nx)$, $a_n=\pi^{-1}\int_\Gamma f(z)\cos nz\,dz$, $b_n=\pi^{-1}\int_\Gamma f(z)\sin nz\,dz$, wobei $f(z)$ in einem komplexen Gebiet das das Intervall $I=(-\pi, +\pi)$ enthält erklärt sei, und Γ ein die Punkte $-\pi$ und $+\pi$ verbindender Weg ist. Untersucht wird die Konvergenz und Cesàro-Summierbarkeit dieser Reihen und ihrer konjugierten Reihen für Funktionen $f(z)$, die auf I Pole besitzen. Grob gesprochen folgt für Pole einer Ordnung $\leqq p$ die (C, p)-Summierbarkeit der Reihe. Ferner wird die Summierbarkeit der Reihen, die sich aus den Ableitungen und Integralen von $f(z)$ ergeben, untersucht. *A. Peyerimhoff* (Giessen)

1887:

Yano, Shigeki. Cesàro summability of Walsh-Fourier series. Tôhoku Math. J. (2) **9** (1957), 267–272.

Let $\sigma_n^{(\alpha)}(x)$ denote the nth (C, α) mean of the Walsh-Fourier series of the integrable function f. The author obtains upper bounds for $\int_0^1 \sup_n |\sigma_n^{(\alpha)}(x)|^p dx$ for $0<p$, by using results of Fine [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 588–591; MR **17**, 31]. For $p>1$, the same bounds had been obtained earlier by Paley [Proc. London Math. Soc. (2) **34** (1932), 241–279] and by Sunouchi [Proc. Amer. Math. Soc. **2** (1951), 5–11; MR **12**, 821].

P. Civin (Eugene, Ore.)

1888:

***Jessen, Børge. Some aspects of the theory of almost periodic functions.** Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 305–314. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

This is an expository talk on almost periodic functions. The first part is a very brief description of the theory of Bohr almost periodic functions and the generalizations of Stepanov, Wiener, Weyl and Besicovitch. The second part is a resumé of the known results on the mean motion, $\lim \arg f(\sigma+it)/t$ as $t\to\infty$, of functions $f(s)$ regular and almost periodic in a strip.

P. Hartman (Baltimore, Md.)

1889:

Bononcini, Vittorio E. Alcuni problemi di massimo per le serie multiple di Fourier. Riv. Mat. Univ. Parma **7** (1956), 255–269.

Der Verf. verallgemeinert frühere Ergebnisse von Szász [Amer. J. Math. **61** (1939), 165–177] auf doppelte Fourierreihen. Es gilt z.B. für $f(x, y)\sim\sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty}\gamma_{r,s}\exp\{i(rx+sy)\}$, $|f(x, y)|\leqq 1$ die Beziehung

$$\Big|\sum_{0\leq r\leq 2p}\sum_{0\leq s\leq 2q}\binom{2p}{r}\binom{2q}{s}\gamma_{r+h,s+k}\Big| \leqq \sum_{0\leq r\leq p}\sum_{0\leq s\leq q}\binom{p}{r}^2\binom{q}{s}^2$$

mit Gleichheit für $f(x, y)=\exp\{i\theta\}\exp\{i(h+p)x+(k+q)y\}$.

A. Peyerimhoff (Giessen)

INTEGRAL TRANSFORMS

See also 1871.

1890:

Butzer, P. L. Halbgruppen von linearen Operatoren und das Darstellungs- und Umkehrproblem für Laplace-Transformationen. Math. Ann. **134** (1957), 154–166.

Let $x(t)\in L_p(0, \infty)$, $1\leqq p\leqq\infty$, and set

$$y(\sigma)=\int_0^\infty \exp(-\sigma t)x(t)\,dt.$$

The Abel sum of the Laguerre series, namely, $l(t, \zeta)=\sum l_n L_n(t)\exp(-\zeta n)$, where $L_n(t)$ is the Laguerre function of order n, $l_n=\int_0^\infty x(t)L_n(t)\,dt$, and $\zeta>0$, can also be thought of as an operator on $y(\sigma)$ with l_n equivalently defined as $l_n=\sum_{k=0}^n \binom{n}{k}(k!)^{-1}y^{(k)}(\frac{1}{2})$. Necessary and sufficient conditions on $l(t, \zeta)$ are obtained in order that $y(\sigma)$ be the Laplace transform of a function $x(t)$ in $L_p(0, \infty)$ for each p, $1\leqq p\leqq\infty$. The author also considers the semigroup of operators $[T(\zeta)x](t)=l(t, \zeta)$ with infinitesimal generator $Ax=tx''+x'+(\frac{1}{2}-\frac{1}{4}t)x$, and, following Hille [see Hille and Phillips, Functional analysis and semigroups, American Mathematical Society, Providence, R.I., 1957; MR **19**, 664; Theorem 10.7.2], connects the order of approximation of $T(\zeta)x$ to x as $\zeta\to 0+$ with x being (i) nullified by A and (ii) in the domain of A. Hermite series are treated in a similar way.

R. S. Phillips (Los Angeles, Calif.)

1891:

Taberski, R. On singular integrals. Ann. Polon. Math. **4** (1958), 249–268.

This paper contains a large number of results similar to some obtained by I. P. Natanson and others. [See, e.g., Natanson, Dokl. Acad. Nauk SSSR (N.S.) **54** (1946), 11–13; **72** (1950), 11–14; MR **8**, 577; **11**, 655.] For $f\in C_{2\pi}$, the class of 2π-periodic continuous functions, the first and the second modulus of continuity are denoted by

$$\omega_1(\delta)=\omega_1(\delta; f)=\max_{0\leq h\leq\delta}\|f(x)-f(x+h)\|,$$

$$\omega_2(\delta)=\omega_2(\delta; f)=\max_{0\leq h\leq\delta}\|f(x-h)-2f(x)+f(x+h)\|,$$

where $\|f(x)\|=\max_{-\pi\leq x\leq\pi}|f(x)|$. The class of functions f for which $\omega_k(\delta; f)\leqq M\delta^\alpha$ $(0<\alpha\leqq k, M=\text{const})$ for $0\leqq\delta\leqq\pi$ is denoted by $\mathrm{Lip}_M^{(k)}\alpha$ $(k=1, 2)$. Consider de la Vallée-Poussin's integral

$$V_n(x)=V_n(x; f)=\frac{(2n)!!}{(2n-1)!!}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos^{2n}\frac{t-x}{2}\,dt.$$

It is shown that (*) $\|V_n(x)-f(x)\|<5\omega_2(n^{-\frac{1}{2}})$. That the order of this approximation is best possible is ensured by showing that for every $\alpha\in(0, 2\rangle$ the following asymptotic formula holds:

$$\sup_{f\in L\mathrm{Lip}_M^{(2)}\alpha}\|V_n(x)-f(x)\|=\frac{2^\alpha M}{\sqrt{\pi}}\Gamma\Big(\frac{1+\alpha}{2}\Big)\frac{1}{\sqrt{n^\alpha}}+o\Big(\frac{1}{\sqrt{n^\alpha}}\Big).$$

For the special cases $\alpha=1, 2$ the correction term is found more precisely tobe of the orders $n^{-3/2}$ and n^{-2} respectively. Results of a similar nature are obtained for Fejér's and Poisson's integrals. It is also shown that if $[f(x_0+h)-f(x_0-h)]/(2h)$ tends to a finite limit as $h\to 0+$, then $V_n'(x_0)$ tends to this limit as $n\to\infty$. This result was previously obtained by Natanson [Konstruktivnaya teoriya funkcii, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949; MR **11**, 591] under the additonal assumption of the existence of $f'(x_0)$.

For de La Vallée-Poussin's integral a result similar to (*) is obtained involving the integral modulus of $f \in L_{2\pi}^p$.

The class $C^{(2)}$ consists of all functions f continuous in $(-\infty, \infty)$ and satisfying the condition that for every $\varepsilon>0$ there exists a number $h_\varepsilon>0$ such that for all x and $0\leq h\leq h_\varepsilon$ we have $|f(x-h)-2f(x)+f(x+h)|<\varepsilon$. The Weierstrass integral is denoted by

$$W_n(x)=W_n(x; f)=\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(t)e^{-n^2(t-x)^2}dt.$$

Results similar to those for $V_n(x)$ with $f \in C_{2\pi}$ are obtained for $W_n(x)$ with $f \in C^{(2)}$.

The paper also contains some results concerning general even non-negative kernels.

J. G. Herriot (Stanford, Calif.)

1892:

Bochner, Salomon; and Kawata, Tatsuo. A special integral transformation in euclidean space. Ann. of Math. (2) **68** (1958), 150–158.

The authors consider limits of the form

$$\lim_{T\to\infty} T^k \int_{E_k} f(x_1, \cdots, x_k)K(Tx_1, \cdots, Tx_k)dx_1\cdots dx_k,$$

where E_k is the k-dimensional Euclidean space, and the kernel K is of the form

$$K(x_1, x_2, \cdots, x_k)=K_0(x_1+\cdots+x_k)\prod_{j=1}^{k} K_j(x_j),$$

each K_i $(i=0, 1, \cdots, k)$ being the Fourier transform of a bounded function in $L_1(-\infty, \infty)$.

A typical result is that the above limit has the value $C_k(2\pi)^k f(0, 0, \cdots, 0)$, where C_k is a constant depending on k, if f is continuous and in $L_1(E_k)$ and if the Fourier transform of f is in $L_1(E_k)$. The techniques of proof are fairly standard Fourier transformation techniques.

The authors also consider generalizations of those results to kernels whose dependence on T is more complex.

P. G. Rooney (Toronto, Ont.)

1893:

Goldberg, Richard R. Inversions of generalized Lambert transforms. Duke Math. J. **25** (1958), 459–476.

The author considers integral transforms F defined by (1) $F(x)=\int_{0+}^{\infty} K(xt)d\alpha(t)$ or (2) $F(x)=\int_{0+}^{\infty} K(xt)\phi(t)dt$, where (3) $K(t)=\sum_{k=1}^{\infty} a_k e^{-kt}$, and $\int_{0+}^{\infty}=\lim \int_{\varepsilon}^{R}$ $(\varepsilon\to 0, R\to\infty)$. When $a_k=1$ $(k=1, 2, \cdots)$, F is the Lambert transform of α or ϕ, respectively.

If $A=\{a_k\}$, $B=\{b_k\}$, $A\approx B$ is defined to mean

$$\sum_{d|m} a_d b_{m/d}=\begin{cases}1 \text{ for } m=1,\\ 0 \text{ for } m=2, 3, \cdots.\end{cases}$$

The author notes that a unique B exists with this property if and only if $a_1\neq 0$. He defines $A \in C_r^*$ if (i) $a_k\geqq 0$ $(k=1, 2, \cdots)$, (ii) $a_k=O(k^{r-1})$ as $k\to\infty$, (iii) $a_1>0$, and (iv) $A\approx B$ with $b_k=O(k^{r-1})$.

For his inversion theory the author considers only transforms whose kernels K are generated as in equation (3) from sequences $A \in C_r^*$ for some $r>-1$. He shows that if F is defined by (1), K being generated by A, $A\approx B$, then under certain conditions on α,

$$\lim_{p\to\infty}\int_0^t B_{p,u}[F]du=\alpha(t) \quad (0<t<\infty),$$

where

$$B_{p,u}[F]=\frac{(-1)^p}{p!}\left(\frac{p}{u}\right)^{p+1}\sum_{n=1}^{\infty} b_n n^p F^{(p)}\left(\frac{np}{u}\right),$$

and that if F is defined by (2), then under certain conditions on ϕ, $\lim_{p\to\infty} B_{p,t}[F]=\phi(t)$ $(0<t<\infty)$.

The method of proof consists in showing that if, for example, F is defined by (1), K being generated by $A\approx B$, then

$$\sum_{k=1}^{\infty} b_k F(kx)=\int_{0+}^{\infty} e^{-xt}d\alpha(t).$$

An application of the Widder-Post inversion formula for the Laplace transformation to this last series gives $B_{p,u}[F]$.

P. G. Rooney (Toronto, Ont.)

1894:

Kupcov, N. P. On absolute and uniform convergence of Fourier integrals. Mat. Sb. N.S. **42(84)** (1957), 461–478. (Russian)

Let $\varphi_\alpha(x, \lambda)$, with $\varphi_\alpha(0, \lambda)=\sin\alpha$, $\varphi_\alpha'(0, \lambda)=-\cos\alpha$, be a solution of $y''(x)-q(x)y(x)+\lambda y(x)=0$, where the real-valued function $q(x)$ is integrable and square integrable on every finite segment of $[0, \infty)$. Let

$$F(\lambda)=\text{l.i.m.}_{n\to\infty}\int_0^n f(x)\varphi_\alpha(x, \lambda)dx$$

be the Fourier transform of $f(x) \in L_2(0, \infty)$ and consider the "Fourier integral" $\int_{-\infty}^{+\infty} F(\lambda)\varphi_\alpha(x, \lambda)d\rho(\lambda)$, where the nondecreasing function $\rho(\lambda)$ satisfies the relation $\int_0^\infty f^2(x)dx=\int_{-\infty}^{+\infty} F^2(\lambda)d\rho(\lambda)$ (at least one such function exists). If the Fourier integral converges absolutely and uniformly on every I, then it coincides with the function $f(x)$. Levitan and Marchenko solved satisfactorily the problem of uniform convergence of Fourier integrals on every I. The author establishes conditions which insure uniform and absolute convergence on every I and conditions which insure uniform convergence on the whole of $[0, \infty)$.

M. Loève (Berkeley, Calif.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 2205, 2206.

1895:

Evgrafov, M. A. Evaluation of the growth of a solution of an integral equation of Volterra type. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 297–302. (Russian)

The author is concerned with estimating the growth, as $x\to\infty$, of solutions of the integral equation $f(x)=\varphi(x)+\int_0^x f(t)K(x, t)dt$, $0\leq x<\infty$, where K satisfies $\int_t^A |K(x, t)|dx<\infty$, $A<\infty$. A typical result, for a related integral equation, runs as follows: suppose that the kernel $K(x, t)$ satisfies the following conditions: (1)

$$\sup_{x\geq h}\int_0^h |K(x-s, x)|ds=\theta<1, \quad h>0;$$

(2) for arbitrary $0<r<r_0$ the series

$$S(r)=\sum_{k=0}^{\infty}\lambda^{k+1}\gamma_1(n)\cdots\gamma_k(r), \quad \lambda=1/(1-\theta),$$

$$\gamma_{k+1}(r)=\sup_{x\geq kh}\int_0^x |K(x-s, x)|e^{sr}ds,$$

is convergent, (3)

$$\int_0^\infty |\varphi(x)|e^{-xr}dx<\infty, \quad r>0;$$

then the integral equation $f(x)=\varphi(x)+\int_x^\infty K(x, t)f(t)dt$, $x>0$, has a unique solution $f(x)$ such that $\int_0^A |f(x)|dx<\infty$

for every finite A, further,

$$\int_0^\infty |f(x)|e^{-xr}dx \leqq S(r)\int_0^\infty |\varphi(x)|e^{-xr}dx.$$

The estimates obtained in this paper are the "continuous" analogues of the estimates obtained in the "discrete" case in a previous paper [M. A. Evgrafov, Dokl. Akad. Nauk SSSR **105** (1955), 625–627; MR **17**, 749], and may be employed to obtain results similar to those in that paper.

J. B. Diaz (College Park, Md.)

1896:

Latta, G. E. The solution of a class of integral equations. J. Rational Mech. Anal. **5** (1956), 821–834.

The aim of this paper is to develop a method for obtaining the solution of a certain class of integral equations. This method depends on showing that the required solution satisfies an ordinary differential equation, and setting up the suitable boundary value problem for the differential equation. The author uses some known examples to show that if $\Gamma f=r(x)$, $\Gamma f=\int_a^b K(x-t)f(t)\,dt$ where $K(u)$ satisfies $uLK(u)+MK(u)=0$ (L, M are ordinary differential operators with constant coefficients), then $\Gamma xf=-\int_a^b (x-t)K(x-t)f dt+x\Gamma f$ and $L(\Gamma xf)=M\Gamma f+xL(\Gamma f)$. If uniqueness exists, then the solution of $\Gamma f=r(x)$ for elementary $r(x)$ can be expressed in terms of $\Gamma g=S_1(x)$, where $L(S_1(x))=0$. The author announces a paper concerning higher dimensional cases.

T. Eweida (Zbl **70**, 329)

1897:

Mihlin, S. G. Singular integrals in L_p spaces. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 28–31. (Russian)

In this note the author announces various results concerning singular integral operators on euclidean m-space. Consider the singular integral operator A defined by

$$Au=a(x)u(x)+\int_{E_m}\frac{f(x,\theta)}{r^m}\,u(y)dy$$

and let $\Phi(x,\theta)$ be the symbol of A. Conditions are given on Φ which imply that A is a bounded operator in L_p. The author further considers the equation

$$A_1u=Au+Tu=g(x)\ (g(x)\in L_p(E_m)),$$

where T is a completely continuous operator on $L_p(E_m)$. If $a(x)$ and $f(x,\theta)$ satisfy certain continuity estimates, then A_1 has a normal resolution and the solution space of the above equation is finite dimensional. The proofs of these results are outlined only for the case of $m=2$.

J. J. Kohn (Waltham, Mass.)

1898:

Vekua, N. P. On a system of singular integro-differential equations and its application in boundary problem of conjunction. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 135–147. (Russian)

The author makes use of the notation and concepts introduced in his earlier works [Soobšč. Akad. Nauk. Gruzin. SSR **13** (1952), 9–14; MR **14**, 968]. The problem under consideration is to find two vectors $\varphi(z)$ and $\psi(z)$ whose components are meromorphic functions in a region and satisfy certain boundary conditions on the simple closed curve bounding this region. A particular case of the present problem is equivalent to the Carleman boundary problem considered in the above mentioned work. The problem is reduced to the solution of a particular case of a quite general singular integral equation. Particular cases of the problem considered here are the so-called first and second boundary value problems of the planar theory of elasticity.

H. P. Thielman (Ames, Iowa)

1899:

Ahmedov, K. T. The Cauchy problem for a class of nonlinear equations in functional spaces. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 9–12. (Russian)

The author is interested in equations of the form

$$\frac{du}{dt}=A\left(t,u,\frac{du}{dt}\right)+B(t,u),$$

where u, du/dt, are elements of a Banach space, and A is analytic in t and u, linear in du/dt. The solution u is to satisfy an initial condition $u|_{t=t_0}=u_0$. The particular case

$$\frac{\partial u}{\partial t}=\int_0^1 K(t,x,s,u(s))\,\frac{\partial u}{\partial t}\,ds+f(t,x,u(x))$$

(author's notation) is considered in detail. The author proves by the method of majorants that if one is not a characteristic value of the kernel $K(t_0,x,s,u_0)$, then there is in a neighbourhood of $t=t_0$ a unique holomorphic solution $u(x,t)$, such that $\lim u(x,t)=u_0(x)$.

Conditions are also given under which multiple solutions exist, and under which only singular solutions exist.

R. Finn (Berlin)

1900:

Baumann, Volker. Eine nichtlineare Integrodifferentialgleichung der Wärmeübertragung bei Wärmeleitung und -strahlung. Math. Z. **64** (1956), 353–384.

Nella risoluzione di problemi stazionari, unidimensionali di propagazione del calore con irraggiamento, si incontra la seguente equazione integrodifferenziale

$$(1)\quad E_g\frac{1}{2\pi n^2}=\int_0^\infty\int_0^h k(\lambda)\,\mathrm{sign}(x-t)K_1(k(\lambda)|x-t|)E(\lambda,T(t))dtd\lambda$$
$$-\int_0^\infty\{K_2(k(\lambda)x)E(\lambda,T_0)-K_2(k(\lambda)(h-x))E(\lambda,T)\}d\lambda$$
$$-\frac{\chi}{2\pi n^2}T'(x),$$

dove $T(x)$ e E_g (costante) sono incognite, $k(\lambda)$ una nota funzione non negativa, continua (o continua a tratti) per $0\leqq\lambda\leqq+\infty$, $T_0\geqq 0$, $T_h\geqq 0$, $\chi\geqq 0$, $n>0$, $h>0$ costanti reali;

$$K_0(t)=\mathrm{Ei}(-t)=-\int_t^\infty\frac{e^{-s}}{s}\,ds,\ t>0;$$

$$K_i(t)=-\int_t^\infty K_{i-1}(s)ds,\ i=1,2,\cdots,\ t\geqq 0;$$

$$E(\lambda,T)=\begin{cases}2c_1\lambda^{-5}(e^{c_2/\lambda T}-1)^{-1}, & T\neq 0,\ \lambda\neq 0,\\ 0, & T=0 \text{ o } \lambda=0,\end{cases}$$

con c_1 e c_2 costanti reali. La funzione incognita $T(x)$ va ricercata nella classe delle funzioni continue non negative per $0\leqq x\leqq h$ e per $\chi>0$ differenziabili. Per una assegnata E_g dovrà poi aversi $T(0)=R_0\geqq 0$ e $T(h)=R_h\geqq 0$. Ricordate alcune essenziali proprietà di $K_i(t)$ e di $E(\lambda,T)$, si consegue un teorema di unicità per le soluzioni di (1) con procedimento classico, applicato all'equazione ottennuta da (1) derivandola rispetto ad x. La laboriosa dimostrazione del teorema di esistenza, nel caso $\chi>0$, si basa sul noto teorema del punto unito (di Schauder), applicato ad uno spazio funzionale convenientemente costruito. Nel caso

$\chi=0$ la (1) si riduce ad una equazione integrale non lineare. *G. Sestini* (Zbl **70**, 104)

FUNCTIONAL ANALYSIS

See also 1692, 1747, 1758, 1787, 1788, 1835, 1844, 1872, 1890, 1934, 1935, 1947, 1963, 2023, 2072, 2081, 2207.

1901:

Ellis, H. W. On the MT^*- and λ-conjugates of $\mathfrak{L}^\lambda$ spaces. Canad. J. Math. **10** (1958), 381–391.

Marston Morse and the reviewer [J. Analyse Math. **4** (1954–55), 149–186; MR **17**, 469] have studied vector function spaces for which each element of the dual is represented by a C-measure, values of the dual element being given by integrals with respect to this measure. In the present paper the author lightens this condition to obtain spaces, called MT^* spaces, in which part but not all of the dual is of this integral type. This part is called the MT^*-conjugate of the space, and the principal purpose of the paper is to compare this conjugate with the λ-conjugate for spaces which are both MT^* spaces and L^λ spaces, the vector function spaces introduced and studied by the author and I. Halperin [Canadian J. Math. **5** (1953), 576–592; MR **15**, 439].

These two conjugates are shown to coincide if the locally compact space E, on which the functions comprising the vector space are defined, is countable at infinity. General conditions for the inclusion of one of these conjugates in the other are obtained, and examples are given to show that the MT^*-conjugate need not contain the λ-conjugate. While the λ-conjugate is shown to contain the MT^*-conjugate for a wider class of spaces E than those countable at infinity, the question of whether this inclusion is always true is not resolved.

W. R. Transue (Gambier, Ohio)

1902:

Kawai, Itizo. Locally convex lattices. J. Math. Soc. Japan **9** (1957), 281–314.

A locally convex lattice is a vector lattice with the locally convex Hausdorff topology defined by a family $\{p_\alpha\}$ of semi-norms (=pseudo-norms) which satisfies the condition: if $|y|\leqq|x|$ then, for each α, $p_\alpha(y)\leqq p_\alpha(x)$. The article under review is a study of such spaces.

The author introduces four continuity conditions for the topology T of a locally convex lattice E, of which the following is typical: T is o-continuous if $a_n\downarrow 0$ (i.e., $a_1\geqq a_2\geqq\cdots$ and $\cap\{a_n\}=0$) implies that the sequence $\{a_n\}$ T-converges to 0. One can also define various conjugate spaces of E: the space E' of all T-continuous linear functionals, the space $E^\sim$ generated by all positive functionals, the space $E^\cdot$ of all o-continuous linear functionals (a linear functional f on E is o-continuous if $a_n\downarrow 0$ implies that $\{f(a_n)\}$ T-converges to 0), and so forth, and there are various topologies resulting from possible pairings. The paper contains numerous theorems relating these notions. A locally convex lattice which is an $\mathfrak{LF}$-space of Dieudonné and Schwartz [Ann. Inst. Fourier Grenoble **1** (1949), 61–101; MR **12**, 417] is called an $\mathfrak{LF}$-lattice. A defining sequence of subspaces $\{E_n\}$ of an $\mathfrak{LF}$-lattice E can be so chosen that each E_n is an ideal in E (i.e., $x\in E_n$ and $|y|\leqq|x|$ imply $y\in E_n$). It is proved that an $\mathfrak{LF}$-lattice E is isomorphic to the space of all continuous real valued functions with compact carrier on a locally compact, σ-compact Hausdorff space if and only if there is a sequence $\{e_n\}$ of positive elements of E such that for each element x of E there is an e_n and a real number $\lambda>0$ such that $|x|\leqq\lambda e_n$.

I. Namioka

1903:

Weston, J. D. Discontinuous linear functionals. Proc. Cambridge Philos. Soc. **54** (1958), 559–560.

Let X be a real vector space having a non-zero linear functional which is continuous with respect to any norm on X. Then X is finite-dimensional.

C. Davis (Providence, R.I.)

1904:

Klonecki, W. On the notion of uniform convergence with respect to a fundamental set of functionals, and its application. Colloq. Math. **5** (1957), 79–84.

Let X be a Banach space with conjugate X^*. A set $\Gamma\subset X^*$ is a fundamental set if there exist positive constants a, K such that for every $\xi\in\Gamma$ and $x\in X$, $\sup|\xi x|\geqq a\|x\|$ if $\|\xi\|\leqq K$. The set Γ_0 is strongly fundamental if $\sup_n|\xi x_n|<\infty$ for every $\xi\in\Gamma_0$ implies $\limsup_n\|x_n\|<\infty$. If one considers vector valued functions defined from a real interval or from a rectangle one can define uniform continuity, differentiability and convergence with respect to a fundamental set Γ in an obvious manner. Various results are obtained concerning the relationship between uniform properties with respect to a fundamental set Γ and strong properties. For example, for a function defined on a real interval, uniform continuity and differentiability with respect to a fundamental set Γ are shown to be equivalent to the respective strong properties. Various applications are given.

R. E. Fullerton (College Park, Md.)

1905:

Everitt, W. N. Inequalities for Gram determinants. Quart. J. Math. Oxford Ser. (2) **8** (1957), 191–196.

Let E, F be measurable linear sets with $E\subset F$. If n complex-valued L^2 functions $f_1, f_2, \cdots, f_n$ on F are linearly independent over E, then

$$\frac{\Delta(f_1,\cdots,f_{n-1};E)}{\Delta(f_1,\cdots,f_{n-1},f_n;E)}\geqq\frac{\Delta(f_1,\cdots,f_{n-1};F)}{\Delta(f_1,\cdots,f_{n-1},f_n;F)}.$$

Here $\Delta(f_1,\cdots,f_n;E)$ denotes the Gram determinant whose elements are $\int_E f_i(x)\overline{f_j(x)}dx$ $(i,j=1,2,\cdots,n)$. This inequality is strict if (but not only if) f_n is not linearly dependent on $f_1,\cdots,f_{n-1}$ over $F-E$. The paper also contains a new proof of the known inequality $\Delta(f_1,\cdots,f_n;E)\leqq\Delta(f_1,\cdots,f_n;F)$ [R. Courant and D. Hilbert, Methods of mathematical physics, I, Interscience, New York, 1953; MR **16**, 426; p. 107].

Ky Fan (Notre Dame, Ind.)

1906:

Sz.-Nagy, Béla. Note on sums of almost orthogonal operators. Acta Sci. Math. Szeged **18** (1957), 189–191.

The author gives a very simple proof of the following theorem due to the reviewer [Revista Mat. Cuyana **1** (1955), 41–55; MR **18**, 219], whose proof was based on a rather complicated combinatorial problem: Let $T_1+\cdots+T_n$ be a sum of permutable Hermitian operators on Hilbert space, satisfying the conditions $\|T_i\|\leqq 1$, $\|T_iT_j\|\leqq\varepsilon^{|j-i|}$, where $0\leqq\varepsilon<1$. Then $\|T_1+\cdots+T_n\|\leqq c(\varepsilon)$. The author gives in addition the estimate $(1+\varepsilon)/(1-\varepsilon)\leqq c(\varepsilon)\leqq(1+\varepsilon^2)/(1-\varepsilon)$. The proof is based on the following numerical lemma: if $s=v_1+\cdots+v_n$ is a sum of real numbers such that $|v_i|\leqq 1$, $|v_iv_j|\leqq\varepsilon^{|j-i|}$, then $|s|\leqq c(\varepsilon)$.

M. Cotlar (St. Louis, Mo.)

1907:

Fenyő, István. Elements of the theory of distributions. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 231–248. (Hungarian)

This is a good exposition of L. Schwartz's theory of distributions [Théorie des distributions, I, II, Hermann, Paris, 1950, 1951; MR **12**, 31, 833] with some modifications due to the present author. *J. Aczél* (Debrecen)

1908:

Vasilach, Serge. Sur la caractérisation de la transformation de Fourier des distributions. C. R. Acad. Sci. Paris **246** (1958), 2836–2838.

Notation: R^n is euclidean n space; $x=(x_1, \cdots, x_n)$, $y=(y_1, \cdots, y_n)$ are coordinates on R^n; D_x $[D_{xy}]$ is the space of C^∞ functions of compact support on R^n $[R^{2n}]$; D_x' $[D_{xy}']$ is the dual of D_x $[D_{xy}]$.

A distribution $K \in D_{xy}'$ defines a continuous linear transformation of $D_x \to D_y'$ and conversely every such transformation is defined by a $K \in D_{xy}'$ by the Schwartz kernel theorem. The author proves that if K satisfies $K \cdot \partial\varphi/\partial y_j = (ix_j)K \cdot \varphi$ for all $\varphi \in D_y$ then

$$K = L_x \exp(-i\sum x_j y_j)$$

where $L_x \in D_{xy}'$ is independent of y. Suppose in addition that K defines a weakly continuous map of $E_y' \to D_x'$ (here E' is the space of distributions of compact support) and $K \cdot \delta_y = 1$. Then $K = \exp(-i\sum x_j y_j)$.

L. Ehrenpreis (Waltham, Mass.)

1909:

Pisanelli, D. Some analytic functionals and their regions of definition. Bol. Soc. Mat. São Paulo **9** (1954), 1–66 (1957). (Portuguese)

Il s'agit d'une mise au point de la théorie des fonctionnelles analytiques, ayant pour but principal l'étude développée des fonctionnelles multilinéaires et des fonctionnelles homogènes qui en dérivent. L'A. suit l'orientation de Pellegrino et Del Pasqua, qui ont cherché à tirer profit de la topologie définie par Fantappiè dans son espace $\mathscr{S}$ des fonctions localement analytiques. Le reviewer ne peut pas s'abstenir de rappeler ce qu'il a montré à plusieurs reprises: cette topologie soulève des difficultés inutiles et des anomalies sérieuses. A la suite de travaux de Köthe, Silva Dias, etc., la systématisation "locale" de la théorie des fonctionnelles analytiques, proposée par le reviewer, a été complètement subordonnée à la théorie des espaces localement convexes, avec des avantages évidents. D'autre part, Tillmann a réussi à définir dans $\mathscr{S}$ une topologie raisonnable d'espace métrique, qui permet de construire une théorie "globale" des fonctionnelles analytiques, parfaitement satisfaisante [#1911 ci-dessous]. Il semble donc inutile d'insister sur une orientation périmée. Toutefois l'A. présente des résultats nouveaux qui sont peut-être indépendants de la forme choisie: il serait alors intéressant de les formuler sous une forme plus commode. *J. Sebastião e Silva* (Lisbonne)

1910:

Pisanelli, D. Caratterizzazione delle regioni quasi lineari. Bol. Soc. Mat. São Paulo **9** (1954), 67–71 (1957).

L'A. donne ici une nouvelle démonstration, plus directe, d'un résultat de Pellegrino et de Del Pasqua, concernant la topologie de l'espace $\mathscr{S}$ de Fantappiè [voir ci-dessus #1909].

J. Sebastião e Silva (Lisbonne)

1911:

Tillmann, Heinz Günther. Die Fortsetzung analytischer Funktionale. Abh. Math. Sem. Univ. Hamburg **21** (1957), 139–193.

Soit Ω la sphère de Riemann. On appelle fonction localement analytique toute fonction $f(z)$ définie dans un ouvert D_f de Ω et analytique en tout point de D_f. Fantappiè avait bâti sa théorie des fonctionnelles analytiques sur l'espace $\mathscr{S}$ des fonctions f localement analytiques (telles que $f(\infty)=0$ si $\infty \in D_f$), muni d'une topologie que l'on peut définir, en prenant pour voisinage $U_\varepsilon(f)$ de chaque $f \in \mathscr{S}$ l'ensemble des fonctions $g \in \mathscr{S}$ telles que: si $d(z, \mathbf{C}D_f) > \varepsilon$, alors $z \in D_g$ et $|f(z)-g(z)| < \varepsilon$.

Mais cette topologie (non séparée) présentait certains inconvénients signalés par le rapporteur [Portugal Math. **9** (1950), 1–130; **12** (1953), 1–47; MR **11**, 524; **14**, 656], qui a systématisé la "théorie locale" des fonctionelles analytiques, en définissant, dans chaque "région linéaire" (A) (constituée par les $f \in \mathscr{S}$ telles que $D_f \supset A$, où A est un vrai sous-ensemble fermé et non vide de Ω), une relation naturelle d'équivalence; alors le quotient du sous-espace (A) de $\mathscr{S}$ par cette relation d'équivalence est déjà un espace vectoriel, $\mathscr{H}(A)$, muni d'une topologie séparée plus maniable; Köthe et d'autres auteurs ont reconnu dans $\mathscr{H}(A)$ un espace localement convexe [Math. Z. **57** (1952), 13–33; MR **14**, 563].

Mais il restait encore à faire l'étude convenable du prolongement analytique des fonctionnelles, ce qui impliquait une "théorie globale". La topologie de Fantappiè ne permettait pas, par exemple, de généraliser le principe des identités. Or l'auteur introduit une nouvelle topologie dans l'espace $\mathscr{S}$, en prenant pour voisinage $V_\varepsilon(f)$ de $f \in \mathscr{S}$ l'ensemble des $g \in U_\varepsilon(f)$ (voir déf. ci-dessus) vérifiant la condition supplémentaire: $d(z, \mathbf{C}D_g) > \varepsilon$ implique $z \in D_f$. Alors on a ce résultat remarquable: la fonction $d(f, g) = \inf\{\varepsilon: g \in V_\varepsilon(f)\}$ est une distance, et on a $V_\varepsilon(f) = \{g: d(f, g) < \varepsilon\}$. L'espace $\mathscr{S}$ muni de cette métrique est séparable; son complété s'identifie à l'espace $\hat{\mathscr{S}} = \mathscr{S} \cup \{\emptyset\}$, où $\emptyset$ est la "fonction" de domaine vide. Une suite (f_n) converge vers g dans $\mathscr{S}$ si: 1) le domaine de f_n converge vers D_g (au sens d'une métrique naturelle); 2) $f_n(z)$ converge vers $g(z)$ unif. sur toute partie fermée de D_g. Le passage au quotient dans chaque région (A) redonne la topologie de $\mathscr{H}(A)$ déjà définie: la "théorie locale" subsiste donc, avec sa forme simplifiée.

Une application F d'un ouvert $\mathscr{G}$ de $\hat{\mathscr{S}}$ dans un espace loc. convexe E est analytique si: 1) $F(\hat{f}) = F(f)$ pour tout prolongement $\hat{f} \in \mathscr{G}$ de f; 2) $F(f_\lambda)$ est une fonction analytique de λ, pour toute "ligne analytique f_λ" (c.-à.-d., fonction "analytique" de λ à valeurs dans $\mathscr{G}$). Maintenant, toute la théorie classique du prolongement analytique (en particulier le principe des identités) se généralise aux fonctionnelles analytiques, sans aucune des anomalies inhérentes à l'ancienne topologie. (L'auteur emploie ici un th. démontré par le rapporteur: toute fonctionelle analytique est continue.) Les prolongements analytiques des fonctionnelles linéaires F se ramènent aux prolongements analytiques de leurs "indicatrices", $\varphi(\lambda) = F_x[(\lambda - x)^{-1}]$.

Enfin, l'auteur généralise tous ces résultats au cas des fonctions $f(p)$ de n variables complexes localement analytiques dans des variétés analytiques complexes, et encore au cas de fonctionnelles de plusieurs arguments.

J. Sebastião e Silva (Lisbonne)

1912:

Fet, A. I. The space of analytic functions and its application to the Cauchy-Kovalevskaya problem. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 215–222. (Russian)

L'A. introduit dans l'espace des fonctions analytiques dans le disque $|z|<1$, une structure d'espace K (de Kantorovitch), par la relation d'ordre suivante: si $f(z)=\sum_{k=0}^{\infty}(\alpha_k+i\beta_k)z^k$, on écrit $f>0$ si $\alpha_k\geqq 0$, $\beta_k\geqq 0$ et au moins l'un de ces coefficients est >0. La convergence en ordre est alors équivalent à la convergence uniforme sur tout compact. Dans l'espace ainsi construit, le problème de Cauchy-Kowalevska pour les équations aux derivées partielles, rentre dans le schème donné par L. V. Kantorovitch, pour la méthode des approximations successives.

G. Marinescu (Zbl **70**, 88)

1913:

Šragin, I. V. On certain operators in generalized Orlicz spaces. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 40–43. (Russian)

Let L_M and L_{M_1} be Orlicz spaces on a measurable set B. Two theorems concerning operators from L_M into L_{M_1} are presented, along with outlines of proofs. (1) A necessary and sufficient condition on a function $b(x)$ $(x\in B)$ is given in order that the multiplication operator $m(x)\to b(x)m(x)$ shall map L_M into L_{M_1}. It follows, as a corollary, that all such operators are continuous. (2) Let $g(u, x)$ be measurable in x $(\in B)$ for each fixed real u and continuous in u for almost all x, and suppose that $M(2u)\leqq \text{const}\, M(u)$ for $u\geqq 0$. The operator $m(x)\to g(m(x), x)$ is weakly continuous from L_M into L_{M_1} if and only if $g(u, x)=a(x)+b(x)u$, where a and b are suitably restricted functions.

M. Jerison (Princeton, N.J.)

1914:

Szökefalvi-Nagy, Béla. On weakly convergent sequences of normal transformations in Hilbert space. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 295–303. (Hungarian)

[Editor's note: The French version of this article has already been reviewed: Acta Math. Acad. Sci. Hungar. **8** (1957), 295–302; MR **20**#223.]

Let $\mathfrak{B}$ be the set of bounded linear operators on a Hilbert space $\mathfrak{H}$, of arbitrary infinite dimensionality. For B_n, $B\in\mathfrak{B}$ write $B_n\to B$ if $(B_nf, g)\to(Bf, g)$ for any f, $g\in\mathfrak{H}$. Then for a contraction $T\in\mathfrak{B}$, so that $\|T\|\leqq 1$, there is a sequence of unitary transformations $U_k\to T$ as $k\to\infty$; the same result with a weaker topology is due to P. R. Halmos [Summa Brasil. Math. **2** (1950), 125–134; MR **13**, 359]. Moreover the U_k can be chosen so that they are unitarily equivalent, and so that $U_k^n\to T^n$ for $n=1, 2, \cdots$. There is a similar result on the approximation to a semi-group of contractions $T(s)$, weakly continuous in s and such that $T(s)\to I$ as $s\to 0$, by a sequence $U_k(s)$, strongly continuous in s. The corresponding result for the approximation to self-adjoint transformations A by sequences of projections E_k is given in terms of resolutions $E_k(\lambda)$ of the identity. Finally the author gives necessary and sufficient conditions for an $S\in\mathfrak{B}$ to be subnormal, i.e., to admit a bounded normal extension on to an $\mathfrak{H}'\supset\mathfrak{H}$; there must be an approximating sequence of normal operators $N_k\to S$, the approximation holding also for powers and for adjoints, so that $N_k^{*m}N_k^n\to S^{*m}S^n$.

F. V. Atkinson (Canberra)

1915:

Ghika, Al. Généralisation des familles spectrales aux transformations linéaires d'un espace hilbertien dans un autre. Com. Acad. R. P. Romîne **7** (1957), 759–764. (Romanian. Russian and French summaries)

Let H_1, H_2 be two Hilbert spaces and W a partially isometric mapping of H_1 into H_2. A linear mapping P of H_1 into H_2 is a W-projection if W^*P is a projection and if $P=WW^*P=PW^*W$. A W-spectral repartition (W-s.r.) is a mapping $t\to W_t$ of R_+ into the set of W-projections satisfying the usual properties of a spectral repartition on R_+ (the multiplication is defined by: $A\circ B=AW^*B$). The construction and various properties of an operational calculus corresponding to a W-s.r. are indicated. It is also stated that for every closed linear mapping T of D_T $(\bar{D}_T=H_1)$ into H_2 there exists a W-s.r. $t\to W_t$ (W depends on T) such that $(Tx|y)=\int_0^\infty td(W_tx|y)$ for every $x\in D_T$ and $y\in H_2$. *C. T. Ionescu Tulcea* (New Haven, Conn.)

1916:

Ghika, Al. Algèbres de transformations linéaires continues d'un espace hilbertien dans un autre. Com. Acad. R. P. Romîne **7** (1957), 831–834. (Romanian. Russian and French summaries)

Let H_1, H_2 be two Hilbert spaces and W a partially isometric mapping of H_1 into H_2. Endowed with the multiplication $(A, B)\to AW^*B$ the Banach space $L(H_1,H_2)$ becomes a Banach algebra. It is stated in this note that if $T\in L(H_1, H_2)$, if W^*T is hermitian and positive and if $T=WW^*T=TW^*W$, then the subalgebra spanned by W and T is isomorphic with the algebra $C([0, \|T\|])$. Various properties of this isomorphism are also given.

C. T. Ionescu Tulcea (New Haven, Conn.)

1917a:

Silverman, R. J. Invariant means and cones with vector interiors. Trans. Amer. Math. Soc. **88** (1958), 75–79.

1917b:

Silverman, R. J.; and Yen, Ti. Addendum to "Invariant means and cones with vector interiors". Trans. Amer. Math. Soc. **88** (1958), 327–330.

The principal theorem is: Let G be a semi-group with an invariant mean, and let V be a vector lattice which is complete in the sense that a non-void subset which has an upper bound has a supremum. Let X be a subspace of an ordered linear space Y such that for each y in Y there is x in X with $y\geqq x$, and let G be represented by linear, monotone transformations of Y into itself which leave X invariant (that is, for g in G, $y\to gy$ is linear and monotone, and if $x\in X$ then $gx\in X$). Then, if f is a monotone linear function on X to V which is G-invariant (that is: $f(x)=f(gx)$), there is a monotone linear G-invariant extension f whose domain is Y.

The first paper proves a weaker form of this theorem (it is assumed that the vector lattice has a unit); in the addendum the theorem as quoted is established, and it is shown that this extension property, for all V, characterizes semi-groups with invariant means.

Finally, a theorem of Krein and Rutman on the existence of a simultaneous eigen-vector for the adjoints of a semi-group of transformations is generalized.

J. L. Kelley (Berkeley, Calif.)

1918:

Zaidman, S. Sur la perturbation presque-périodique des groupes et semi-groupes de transformations d'un espace de Banach. Rend. Mat. e Appl. (5) **16** (1957), 197–206.

Let $[G(t); -\infty<t<\infty]$ be a group of contraction operators on a Banach space X and let A be its infinitesimal generator. Suppose further that $G(t)x$ is strongly almost periodic for each $x\in X$. Then $y(t)=\int_0^t G(t-u)x(u)du$ is almost periodic whenever $x(t)$ is almost periodic, and any solution of $dx/dt=Ax+f(t)$ contained in a compact subset of X is also almost periodic provided that $f(t)$ is strongly continuously differentiable and almost periodic. Similar results are obtained for a semi-group of operators $[S(t); t\geqq 0]$ where $\|S(t)\|\leqq\exp(\beta t)$ and $\beta<0$.

R. S. Phillips (Los Angeles, Calif.)

1919:

Hufford, George. One-parameter semi-groups of maps. Duke Math. J. **24** (1957), 443–453.

A semi-group of maps $[u_t; t\geqq 0]$ on a space E into itself satisfies the property $u_s\circ u_t=u_{s+t}$ where the $\circ$ corresponds to the operation of composition; it is also assumed that u_0 is the identity map. Associated with such a family of maps is the semi-group of translation operators $[T_t; t\geqq 0]$ defined as $(T_tf)(x)=f[u_t(x)]$. Suppose now that E is a compact Hausdorff space and C is the Banach algebra of continuous real-valued functions on E with $\|f\|=\sup|f(x)|$. Let $[u_t]$ be a semi-group of continuous maps on E with the property $\lim_{t\to 0} u_t(x)=x$ for each x in E, and let $[T_t]$ be the associated semi-group of translation operators acting on C. Then $[T_t]$ is strongly continuous; and the resulting infinitesimal generator Ω has as its domain a subalgebra of C containing the element 1, and satisfies the relation $\Omega(f\cdot g)=\Omega f\cdot g+f\cdot\Omega g$. Conversely, if Ω is the infinitesimal generator of a strongly continuous semi-group of operators on C, if its domain contains the element 1 and satisfies the above relation, then it is the generator of a translation semi-group and hence defines a unique semi-group of maps $[u_t]$ continuous on $E\times[0,\infty)$ to E. If the $[u_t]$ satisfy the conditions stated in the first part of the above result and E is merely completely regular, then it is possible to define a compactification of E and an extension of the $[u_t]$ so that the extended maps retain these properties. The author also considers various extensions of the converse theorem.

R. S. Phillips (Los Angeles, Calif.)

1920:

***Yosida, Kôsaku. Semi-group theory and the integration problem of diffusion equations.** Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 405–420. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

This is an expository paper dealing with recent contributions to the theory of semi-groups of linear operators and their applications to the diffusion equation. Most of the discussion centers about the results of Yosida [Osaka Math. J. **5** (1953), 65–74; Proc. Japan Acad. **30** (1954), 19–23, 273–275; MR **15**, 36; **16**, 370] on the temporally inhomogeneous diffusion equation defined on a connected domain of a Riemannian space. The researches of W. Feller [Ann. of Math. (2) **55** (1952), 468–519; MR **13**, 948] and E. Hille [J. Analyse Math. **3** (1954), 81–196; MR **16**, 45] for the one-dimensional temporally homogeneous case, and those of E. Hille [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 20–25; MR **15**, 706] and T. Kato [J. Math. Soc. Japan **6** (1954), 1–15; MR **15**, 965] for the temporally homogeneous Kolmogoroff equations, are also described. The concluding section deals with the more abstract approaches of T. Kato [ibid. **5** (1953), 208–234; MR **15**, 437] and R. S. Phillips [Trans. Amer. Math. Soc. **74** (1953), 199–221; MR **14**, 882] to the temporally inhomogeneous processes.

R. S. Phillips (Los Angeles, Calif.)

1921:

Arens, Richard. Inverse-producing extensions of normed algebras. Trans. Amer. Math. Soc. **88** (1958), 536–548.

Let A be a commutative normed algebra with an identity element. A normed algebra B which contains A as a subalgebra is called a "superalgebra" of A. The fundamental result in the present paper is that, if c is any element of A which is not a topological divisor of zero (i.e. $ca_n\to 0$ implies $a_n\to 0$), then there exists a superalgebra B of A in which c is regular (i.e. possesses a multiplicative inverse). This result was obtained by Šilov [Mat. Sb. N.S. **21(63)** (1947), 25–47; MR **9**, 445] for A a Banach algebra with c as a generator. The construction of the superalgebra in the general case is made as follows: First denote by t_0 the infimum of $\|ca\|$, for $a\in A$ with $\|a\|=1$. For $t\geqq t_0$, let $A(x;t)$ be the class of all formal power series $f(x)=a_0+a_1t+a_2t^2+\cdots$ such that $\|f(x)\|=\sum\|a_n\|t^n<\infty$. Then $A(x;t)$ is a normed algebra which is a superalgebra of A. Denote by J the smallest closed ideal in $A(x;t)$ which contains $1-cx$ and denote the quotient algebra $A(x;t)/J$ by $A(t)$. Then $A(t)$ is the desired superalgebra, the inverse of c being the element $x-J$, which, moreover, has norm equal to t. In a natural sense, $A(t)$ is the most economical superalgebra in which c has an inverse of norm t.

If $c_1,\cdots,c_n$ is any finite set of non-topological divisors of zero in A, then a superalgebra can be constructed in which each of these elements is regular. The problem of constructing a superalgebra in which simultaneously all non-topological divisors of zero of A become regular was solved for certain special Banach algebras by the reviewer [Duke Math. J. **14** (1947), 1063–1077; MR **9**, 358].

Denote by $\Delta(A)$ the class of all homomorphisms of A onto the complex field. In the usual weak topology, $\Delta(A)$ is a compact Hausdorff space. The space $\Delta(A(t))$ is homeomorphic in a natural way with the subset of $\Delta(A)$ consisting of all $\eta\in\Delta(A)$ such that $t|\eta(c)|\geqq 1$. In terms of this result, conditions that $A(t)$ be semi-simple can be formulated. In the special case in which c generates A, $A(t)$ will be semi-simple provided tt_0 is greater than the spectral radius of c.

A number of open questions, suggested by these results and their proofs, are mentioned.

C. E. Rickart (New Haven, Conn.)

1922:

Kanzaki, Teruo. A note on the primitive ring of continuous linear transformations. Mem. Osaka Univ. Lib. Arts Ed. Ser. B **6** (1957), 11–12.

Let M be a left vector space and N a right vector space over a common division ring D, which are dual relative to the non-degenerate bilinear functional (x, y), $x\in M$, $y\in N$. Denote by $L(M, N)$ the ring of all continuous linear transformations of M (those linear transformations of M whose adjoints map N into itself). If S is a subspace of M [of N], denote by S' the set of all elements y of N [of M] such that $(x, y)=0$ [$(y, x)=0$], for all x in S. A subspace S is closed if, and only if, $S''=S$. The following theorem is proved: Let (M, N) satisfy the condition that for each closed subspace S of M, there exists a closed subspace Q

of M, such that $M=S\oplus Q$ and $N=S'\oplus Q'$. Then an element β in $L(M, N)$ is regular (in the sense of von Neumann) if, and only if, $M\beta$ and β^*N (the ranges of β and its adjoint β^*) are closed subspaces of M and N respectively. Using this result, a new proof is given that $L(M, N)$ is a regular ring if N is the space of all linear functionals on M. It is also shown that $L(M, N)$ is not regular, in the case that M, N are both of countable dimension over D.

K. G. Wolfson (New Brunswick, N.J.)

1923:

Ghika, Alexandru. L'approximation des éléments dans les modules paranormés. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1(49)** (1957), 47–57.

The author continues his work on unitary (left) modules E over a "corpoidal" algebra A (i.e., an algebra A over the real field R with a basis S that forms a multiplicative group whose identity is the identity of A, and such that $S=-S$). (See Com. Acad. R. P. Romîne **5** (1955), 317–323, 503–507, 955–958 [MR **17**, 767] for other definitions.) Let p be a paranorm ($=S$-norm) on E, and let q be a paranorm on A such that $q(\xi\eta)\leqq q(\xi)q(\eta)$ for all $\xi, \eta\in A$, and let E and A carry respectively the topologies induced by p and q. Then E is a topological A-module if and only if there is a positive $\mu\in R$ such that $p(\xi x)\leqq \mu q(\xi)p(x)$ for all $\xi\in A$, and $x\in E$. If this latter holds, then p may be replaced by an equivalent paranorm with $\mu=1$.

By assuming that S is finite and by specializing the nature of q, the author is able to generalize certain classical theorems on normed vector spaces relating weak and strong convergence.

M. Henriksen (Lafayette, Ind.)

1924:

Nakai, Mitsuru. Some expectations in AW^*-algebras. Proc. Japan Acad. **34** (1958), 411–416.

Let M be a B^* algebra with identity, A a commutative AW^* algebra in the center of M, and containing the identity. Definition: A is regularly embedded in M if, whenever $\{e_i\}$ is a set of $\perp$ projections in A with supremum I, and, for some x in M, $e_ix=0$ for all i, then $x=0$.

Theorem: In order that for every nonzero $x\in M$ there be some conditional expectation e in M with range A such that $e(x^*x)\neq 0$, it is necessary and sufficient that A be regularly embedded in M.

Theorem: If A is regularly embedded in M, then the linear space of A-module homomorphisms of M into A which are norm-continuous is spanned algebraically by the quasi-expectations in M with range A (quasi-expectations being defined like conditional expectations, except that they need not send I to I).

One tool used is a Hahn-Banach type theorem for module homomorphisms into the scalars of the module, the boundedness preserved being with respect to a subadditive positive-homogeneous function into the self-adjoint scalars.

J. Feldman (Berkeley, Calif.)

1925:

***Gelfand, I. Some aspects of functional analysis and algebra.** Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 253–276. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

A discursive account of certain aspects of functional analysis which are more or less connected with quantum mechanics. The six sections are entitled: (1) Introduction; (2) Representations of groups; (3) Generalized functions; (4) Generalized functions and representations of groups; (5) Differential operators; (6) Analysis in functional spaces.

In section 2 the irreducible representations of the real and complex unimodular groups are described in semi-heuristic terms and some suggestive remarks are made about how one might generalize the classical finite group theorems regarding the number of inequivalent irreducible representations and the sum of the squares of the dimensions of the latter via suitable "counting of parameters". There are also interesting discussions of the "reasons" for the occurence of the supplementary series and of representation spaces whose elements are analytic functions.

Section 3 begins with a brief history of the subject (of the section title) and then presents an outline of some work of the author and Shilov on a generalization of Schwartz distributions applicable to rapidly growing functions and to a Cauchy problem which involves such functions. [Uspehi. Mat. Nauk (N.S.) **8** (1953), no. 6(58), 3–54; MR **15**, 867.] It concludes with two remarks of a general nature and the formulation of two problems.

Section 4 describes how generalized functions have been used in obtaining the Plancherel formula for Lie groups and indicates how they might be of further use in constructing a theory of spherical functions for non compact subgroups of Lie groups and in certain related questions.

The greater part of section 5 consists of an account of the status of our knowledge of the spectral theory of differential operators, ordinary and partial, self-adjoint and non-self-adjoint with emphasis on important unsolved problems and recent progress. The section concludes with a summary of recent work on the so called "inverse problem" for second order linear equations.

Section 6 consists mainly of a brief account of the (non rigorous) method of integrating the equations of quantum field theory by an integration in function space originally due to Feynman and discovered independently by the author and Minlos [Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 209–212; MR **16**, 100].

G. W. Mackey (Cambridge, Mass.)

1926:

***Harish-Chandra. Representations of semisimple Lie groups.** Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 299–304. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

This is a rapid survey of some of the principal features of the author's work on infinite dimensional representations of semi-simple Lie groups as it stood in 1954. Topics mentioned include the following. (1) The space of all "well behaved" vectors and its density and invariance properties. (2) The finiteness of the multiplicities which occur when a quasi simple representation of a semi-simple Lie group is restricted to the complete inverse image of a maximal compact subgroup of the adjoint group. (3) The type-I-ness of the unitary representations of a semi-simple Lie group. (4) Distributions as characters. (5) The Plancherel formula. (6) The irreducible unitary representations associated with a conjugacy class of Cartan subgroups.

G. W. Mackey (Cambridge, Mass.)

1927:

Dixmier, Jacques. On unitary representations of nilpotent Lie groups. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 985–986.

If G is a locally compact separable unimodular group it is known that the regular representation of G may be expressed as a direct integral of factor representations U_x

where the χ run over a measure space Ω supplied with a positive measure μ. Furthermore if f is in $L_1(G)\cap L_2(G)$ and $U_\chi(f)$ is the operator corresponding to f in the representation U_χ, the Plancherel formula states that

$$(*) \qquad \int_G |f(x)|^2 dx = \int_\Omega \mathrm{tr}(U_\chi(f)^* U_\chi(f)) d\mu(\chi),$$

where dx is Haar measure on G.

The author announces the following results in the case that G is a simply connected nilpotent Lie group. 1) Ω can be chosen to be the complement of the zeroes of a polynomial in a real q-dimensional Euclidean space. 2) There is a fixed Hilbert space H and for each χ in Ω an irreducible representation U_χ of G on H; for each f in $L_1(G)$, the Fourier transform of f defined by $U_\chi(f)=\int_G U_\chi(x)f(x)dx$ is strongly continuous and zero at infinity on Ω. 3) There is a positive measure μ on Ω defined by a regular differential form of degree q so that (*) is valid for f in $L_1(G)\cap L_2(G)$. *K. de Leeuw* (Stanford, Calif.)

1928:

Dixmier, Jacques. Sur les représentations unitaires des groupes de Lie nilpotents. II. Bull. Soc. Math. France **85** (1957), 325–388.

This paper contains the proofs of the results announced in the paper reviewed above.

K. de Leeuw (Stanford, Calif.)

1929:

Dixmier, J. Sur les représentations unitaires des groupes de Lie nilpotents. III. Canad. J. Math. **10** (1958), 321–348.

In the second article of this series [see the above review] general results concerning the unitary representations of simply connected nilpotent Lie groups are obtained. The third article is devoted to the explicit description of these results for groups of dimension 5 or less. This author first constructs all nilpotent Lie algebras of dimension $\leqq 5$. He determines the irreducible unitary representations of the associated simply connected Lie groups and the explicit form of the Fourier transform and Plancherel formula. A consequence of his analysis is the fact that there may be hermitian characters that correspond to more than one irreducible unitary representation or that do not correspond to any irreducible unitary representation. He also shows that for non-simply connected groups, the Plancherel formula must involve exceptional representations. *K. de Leeuw* (Stanford, Calif.)

1930:

Helgason, Sigurður. Topologies of group algebras and a theorem of Littlewood. Trans. Amer. Math. Soc. **86** (1957), 269–283.

Let G be a locally compact group. For any function $f\in L^1(G)$ let L_f be convolution by f on $L^2(G)$ and $\|L_f\|$ the bound of the operator L_f, which the author calls the spectral norm of f. The author studies linear operators on $L^1(G)$ which are spectrally continuous in the sense that they are continuous from the spectral norm topology to the topology of the L^1-norm. Theorem A: If G is compact the spectrally continuous linear operators on $L^1(G)$ which commute with all left translations are precisely right convolutions by L^2-functions. This result yields a sharpening of a well known theorem of Littlewood on Fourier series. The case of commutative groups has previously been treated by A. Grothendieck [C. R. Acad. Sci. Paris **239** (1954), 577–579 and 607–609; MR **16**, 596] and the present author [Ann. of Math. (2) **64** (1956), 240–254; MR **18**, 494].

The non-compact case turns out to be quite different. Assuming G unimodular and separable so that the general Plancherel formula can be used, the author shows (theorem B) that the only spectrally continuous operator which commutes with right translations is 0, provided that G is connected. In the proof of theorem B the author uses the assumption that G be connected twice. First to construct a sufficiently large class of approximate identities in $L^1(G)$ {here connectedness is probably not necessary}. Secondly, to apply the structure theorem of Iwasawa and Yamabe about connected locally compact groups. {The reviewer wonders whether this could not be avoided too.} At any rate the question as to what happens when G is not connected is left open in the non-compact non-commutative case. *F. I. Mautner* (Paris)

1931:

Hausner, Alvin. The tauberian theorem for group algebras of vector-valued functions. Pacific J. Math. **7** (1957), 1603–1610.

A treatment of the L_1-algebra of functions on a locally compact abelian group in which the functions considered have values in a Banach algebra. The norm and product (convolution) are defined as usual, but using the norm and product in the Banach algebra under the integral. The Fourier transform of an element in L_1 is defined as a function on the product space of the dual group and the maximal ideals of the Banach algebra, and this product space is shown to be the space of regular maximal ideals of L_1. It is shown that every proper closed ideal of L_1 is contained in a regular maximal ideal under the assumptions: the Banach algebra of function values is regular, semi-simple, and the elements of it which have compact support (in its maximal ideal space) are dense.

W. Ambrose (Cambridge, Mass.)

1932:

Mirakov, V. E. The majorant principle for Čebyšev's method. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 171–174. (Russian)

Soit $P(x)=0$ une équation entre deux espaces de type B_K [L. V. Kantorovitch. Dokl. Akad. Nauk SSSR (N.S.) **76** (1951), 17–20; MR **12**, 835], majorée par l'équation $Q(z)=0$ (entre les espaces des normes abstraites correspondantes). L'A. applique la méthode de L. V. Kantorovitch, en remplaçant la formule de Newton par la formule

$$x_{n+1}=x_n-[I+\tfrac{1}{2}\Gamma_0 P''(x_n)\Gamma_0 P(x_n)]\Gamma_0 P(x_n)$$

où $\Gamma_0=[P'(x_0)]^{-1}$, P' étant la dérivée Gâteaux de P. On obtient des résultats analogues aux résultats de L. V. Kantorovitch. Dans le cas des espaces normés on a une équation majorante générale de la forme $\frac{1}{6}Mz^3+\frac{1}{2}Kz^2-z/B+\eta/B=0$; les coefficients intervenant dans les conditions $\|\Gamma_0 P(x_0)\|\leqq\eta$, $\|\Gamma_0\|\leqq B$, $\|P''(x)\|\leqq K$, $\|P'''(x)\|\leqq M$.

G. Marinescu (Zbl **71**, 116)

1933:

Krasnosel'skiĭ, M. A.; and Rutickiĭ, Ya. B. Some nonlinear operators in Orlicz spaces. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 363–366. (Russian)

Continuing their study of operators in Orlicz spaces [see same Dokl. **89** (1953), 601–604; MR **15**, 137], the authors present sufficient conditions for Fréchet differentiability of an operator of the form $u(x)\to f(x,u(x))$, defined on a sphere in an Orlicz space, and for complete continuity of an operator of the form $u(x)\to\int K(x,y,u(x))dy$. No proofs are given.

M. Jerison (Princeton, N.J.)

CALCULUS OF VARIATIONS

See also 1838.

1934:

Križanič, Franc F. Sur le lemme fondamental du calcul des variations. Slovenska Akad. Znan. Umet. Razred Mat. Fiz. Tehn. Vede. Ser. A. **7** (1956), 21–33. (Slovenian. French summary)

Le lemme fondamental du calcul des variations et ses généralisations (Zermelo, Haar, Hilbert, Mason, Kubota) sont seulement des cas particuliers d'un théorème général: "Soit A un opérateur linéaire défini dans un domaine $\mathfrak{D}(A)$ partout dense dans un espace hilbertien $\mathfrak{H}$, et soient a_k $(k=1, \cdots, n)$ des éléments linéairement indépendants. Soit $\mathfrak{E}$ l'ensemble de tous les éléments y de $\mathfrak{D}(A)$, orthogonaux aux a_k, donc $(a_k, y)=0$. Chaque élément x avec la condition $(x, Ay)=0$ quel que soit $y \in \mathfrak{E}$, appartient au domaine $\mathfrak{D}(A^*)$ de l'opérateur adjoint A^* et l'on a $A^*x=\sum \lambda_k a_k$."

La démonstration repose sur les trois lemmes. 1) Les y avec la condition $(a_k, y)=0$ forment un sousespace $\mathfrak{H}'$ avec la condition $\mathfrak{H}'+\mathfrak{L}\{a_k\}=\mathfrak{H}$, où $\mathfrak{L}\{a_k\}$ est le complément orthogonal de $\mathfrak{H}'$. 2) Pour tout élément $y \in \mathfrak{H}$ on a $y=y'+\sum \lambda_k b_k$, où $y' \in \mathfrak{H}'$ et $b_k \in \mathfrak{D}(A)$. 3) L'ensemble $\mathfrak{E}$ est partout dense dans $\mathfrak{H}'$.

On démontre aussi théorème 2: "Chaque élément x avec la condition $(x, Ay)=0$ quel que soit $y \in \mathfrak{D}(A)$ avec la propriété $(a_k, Ay)=0$, s'exprime par la somme $x=b+\sum \lambda_k a_k$ où b est une solution de l'équation $A^*b=0$."

D. Rašković (Belgrade)

1935:

Citlanadze, Ė. S. On a conditional extremal problem with a countable number of conditions. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **24** (1957), 71–87. (Russian)

In the Hilbert space $L_2(0, 1)$ the existence of critical values for a functional $f(x)$ on the set N of x subject to the countable set of conditions $\varphi_i(x)=0$ is discussed. At a critical value $L_f x=\sum \lambda_i L_{\varphi_i} x$, where $L_f x=\text{grad } f(x)$. The equations for the λ are equivalent to the set $\sum (L_{\varphi_i} x, L_{\varphi_j} x)\lambda_j(x)=(L_{\varphi_i} x, L_f x)$. The infinite determinant of the left-hand-side expressions is denoted by $\Delta(x)$. On the assumption that $\|\Delta(x)\| \geqq C>0$ throughout N, and assumptions of boundedness and certain Lipschitz conditions on the operators involved, it is shown that the expressions obtained by solving finite segments of the infinite set of equations for the λ tend to limits, $\lambda_i(x)$.

$L_f x$ is not equal to $\sum \lambda_i(x)\, L_{\varphi_i}(x)$ at noncritical points, and the differential equation $dx/dt=L_f x-\sum \lambda_i(x) L_{\varphi_i} x$ determines trajectories through each noncritical point along which $f(x)$ increases, reaching eventually a critical value. Under certain assumptions on the L's and the determinant Δ, a critical value corresponding to each homotopy equivalence class exists.

J. L. B. Cooper (Cardiff)

GEOMETRIES, EUCLIDEAN AND OTHER

See also 1954, 1955, 1956, 1957.

1936:

Kostyuk, A. I. Some questions in the theory of geometric constructions. Luc'kiĭ Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **6** (1958), no. 3, 37–55. (Ukrainian)

1937:

Kostyuk, A. I. On geometric construction in the Riemann plane. Luc'kiĭ Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **6** (1958), no. 3, 57–73. (Ukrainian)

1938:

Metelka, Václav. Über ebene Konfigurationen $(12_4, 16_3)$**, die mindestens einen** D**-Punkt enthalten.** Časopis Pěst. Mat. **82** (1957), 385–439. (Czech. Russian and German summaries)

A configuration $(12_4, 16_3)$ in the complex projective plane consists of 12 points lying by threes on 16 lines. Of the 4 lines through one point, each contains 2 other points. The remaining 3 points are "separated" from the first point. Hence there is a graph of degree 3 whose 12 nodes represent the 12 points while its 18 branches indicate which pairs of points are "separated". The author's brother Josef [Časopis Pěst. Mat. **80** (1955), 133–145; MR **17**, 655] has already enumerated the 8 cases in which the graph contains (as a disconnected piece) a complete 4-point or "tetrahedron"; e.g., the graph for Hesse's $(12_4, 16_3)$ consists of 3 such tetrads (the four A's, four B's and four C's of Coxeter, Bull. Amer. Math. Soc. **56** (1950), 413–455 [MR **12**, 350], p. 440). In the present paper, the author enumerates the 57 further cases in which the graph contains two triangles with a common side. In each case he gives coordinates for all the vertices of a typical specimen of the configuration.

H. S. M. Coxeter (Toronto, Ont.)

1939:

Ede, John D. Rhombic triacontahedra. Math. Gaz. **42** (1958), 98–100.

The icosahedral group is transitive on a set of 30 planes which are the external bisectors of the dihedral angles of a regular icosahedron (or of a regular dodecahedron). There are hundreds of polyhedra whose faces lie in these 30 planes in such a way as to retain icosahedral symmetry. The author proposes a notation which could be developed into a complete classification of these solids in the manner of Coxeter, Du Val, Flather, and Petrie, "The fifty-nine icosahedra" [University of Toronto Press, 1938]. In particular, the ordinary triacontahedron [Coxeter, Regular polytopes, Methuen, London, 1948; MR **10**, 261; p. 25] is A, the compound of 5 cubes [ibid., p. 49] is E, the small stellated triacontahedron [ibid., p. 103] is $Cd_1e_1f_1$, and the great stellated triacontahedron is $Gh_1h_3i_3j_3$.

H. S. M. Coxeter (Toronto, Ont.)

1940:

Nicolae, Radu L. Propriétés fonctionnelles des coniques. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **8** (1956), 695–702. (Romanian. Russian and French summaries)

Le but de cette note est de généraliser une propriété relative à la parabole, déjà mentionnée auparavant par D. Pompeiu. Voici ce que l'auteur obtient: Parmi les courbes planes continues, les paraboles sont les seules à remplir la condition suivante. Etant donnés sur la courbe deux couples de points quelconques (M_1, M_2), (M_3, M_4), l'égalité de la différence de leur coordonnées — calculées dans une direction quelconque sauf dans celle du point à l'infini et à partir d'un axe contenant le point à l'infini — entraîne le parallélisme des droites M_1M_4, M_2M_3. Cette propriété est ensuite étendue aux paraboloïdes et à l'intersection de deux cylindres paraboliques. Enfin pour les coniques dont les coordonnées sont exprimées en fonction de deux trinomes du second degré en t, le parallélisme des droites M_1M_4, M_2M_3, correspondant aux deux couples de points

(M_1, M_2), (M_3, M_4) dont la différence des paramètres respectifs est égale, subsiste. Toutefois, tout en y croyant l'auteur n'a pu démontré si cette dernière propriété caractérise bien les coniques. *F. Semin* (Istanbul)

1941:

Manevič, V. A. Representation of collineation on a plane in the form of the product of two polar correspondences. Ukrain. Mat. Z. **10** (1958), no. 2, 219. (Russian)

1942:

***Lombardo-Radice, L. Questioni algebrico-geometriche relative ai teoremi "$p=0$".** Convegno italo-francese di algebra astratta, Padova, aprile, 1956, pp. 37–47. Edizioni Cremonese, Rome, 1957. vi+72 pp.

In the present lecture, the author begins by recalling the Hall ternary operation $\circ$ (allowing one to represent the lines of any projective plane by an equation of the form $y=x\cdot m\circ b$), and the definition of the characteristic p of a projective plane. Then, by means of simple geometric considerations, he proves that if $p=2$, i.e., if the plane is a Fano plane then $a^h\cdot a^k\circ a^i=a^{h+k}+a^i$ for any a of the coordinate ring and arbitrary positive integers h, k, i. Finally, he shows that any finite projective plane of characteristic $p=3$ has the rank (number of points on a line diminished by one) $r\equiv 0, 1, 3, 9, 12 \bmod 13$.

In the case of finite planes, the result for $p=2$ is included in a theorem obtained since by A. M. Gleason [Amer. J. Math. **78** (1956), 797–807; MR **18**, 593], according to which every Fano plane is Desarguesian.

B. Segre (Rome)

1943:

Ito, Makoto. A note on non-Desargues projective plane. Proc. Japan Acad. **34** (1958), 420–421.

The author gives a proof of the well-known equivalence of the little Desargues theorem and the fourth harmonic point theorem based on the paper of N. S. Mendelsohn [Canad. J. Math. **8** (1956), 532–562; MR **18**, 665].

P. Abellanas (Madrid)

1944:

Fladt, Kuno. Die allgemeine Kegelschnittsgleichung in der ebenen hyperbolischen Geometrie. II. J. Reine Angew. Math. **199** (1958), 203–207.

In einer Abhandlung des gleichen Titels [dasselbe J. **197** (1957), 121–139; MR **19**, 57] wurde erstmals mit elementaren Mitteln eine vollständige Diskussion des hyperbolischen Kegelschnitts

$$f(x)=a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{23}x_2x_3+2a_{31}x_3x_1 +2a_{12}x_1x_2=0$$

durchgeführt. Es fehlten aber noch die in den a_{ik} ganzrationalen Kriterien für das Eintreten der einzelnen Fälle. Diese sollen jetzt, jedoch auf einem ganz anderen Wege, bestimmt werden. *Aus der Einleitung*

CONVEX SETS AND DISTANCE GEOMETRIES

1945:

Abrham, Jaromír. Über die Stabilität von Lösungen im Transportproblem der linearen Programmierung. Czechoslovak Math. J. **8(83)** (1958), 131–138. (Russian summary)

The paper is concerned with the standard Hitchcock Transportation Problem. In describing the results, the abstract algebraic terms used by the author will be replaced by the equivalent graph theoretic ones for the sake of brevity and clarity.

A solution of a transportation problem is called "stable" if it is possible to change all supplies and demands by some positive amount without having to change the set of routes used in a solution. A solution is "simple" if the routes used form a "tree". It is "indecomposable" if this tree is connected. Let $X=(x_{ij})$ be a simple indecomposable solution and let $\rho(X)$ be the smallest of the numbers x_{ij} which is not zero. Let the supplies and demands be changed by amounts $\alpha_1, \cdots, \alpha_m$ and $\beta_1, \cdots, \beta_n$ respectively. Theorem 1: If $\sum|\alpha_i|+\sum|\beta_j|<\rho(X)$, then there exists a solution of the perturbed problem which uses exactly the same routes as the original solution X. Theorem 2 gives a lower bound for the number $\rho(X)$ for non-degenerate problems, i.e., problems in which all solutions are indecomposable. *D. Gale* (Providence, R.I.)

1946:

Peck, J. E. L. Yet another proof of the minimax theorem. Canad. Math. Bull. **1** (1958), 97–100.

The author gives an elegant proof of the minimax theorem which uses no matrices and almost no topology, and makes little use of the geometry of convex sets. The form of the theorem assumes that one of the pure strategy spaces is finite (the corresponding player is shown directly to have an optimal strategy) and no assumption is made about the boundedness of the payoff function. The proof follows very directly from the following lemma: If F^* is the set of convex linear combinations of a finite number of affine functions defined on a convex subset K of a real linear space, then

$$\sup_{x\in K}\min_{f\in F^*} f(x)=\min_{f\in F^*}\sup_{x\in K} f(x).$$

The proof of the lemma involves induction.

H. Raiffa (Cambridge, Mass.)

1947:

Efimov, N. V.; and Stečkin, S. B. Some properties of Čebyšev sets. Dokl. Akad. Nauk SSSR (N.S.) **118** (1958), 17–19. (Russian)

A set M in a Banach space X is a Čebyšev set if to each x in X there exists a unique nearest element of M. Theorem: Among bounded closed sets in a finite dimensional space X, the Čebyšev sets coincide with the convex sets if and only if the unit ball of X is strictly convex and smooth. The major effort is devoted to proving that if the unit ball is smooth, then a Čebyšev set is convex. {Reviewer's remark. The theorem has been proved by H. Busemann [Mat. Tidsskr. B. **1947**, 32–34; MR **9**, 302].} *M. Jerison* (Princeton, N.J.)

1948:

Ohmann, D. Kurzer Beweis einer Abschätzung für die Breite bei Überdeckung durch konvexe Körper. Arch. Math. **8** (1957), 150–152.

Let K, K_i $(i=1, 2, \cdots, m)$ be convex bodies in E_n such that $K\subseteq\bigcup K_i$. Let $b(K;\xi)$ be the width of K corresponding to the direction ξ, and let ξ_i be any set of m independent directions. The author endeavours to prove that $\sum_i[b(K_i;\xi_i)/b(K;\xi_i)]\geqq 1$, but due to some misprints and to an excessive conciseness the reviewer has been unable to follow all the details of the proof.

L. A. Santaló (Buenos Aires)

1949:

Radziszewski, Konstanty. Sur une fonctionnelle définie sur les ovales. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **10** (1956), 57–59 (1958). (Polish and Russian summaries)

In this paper the author proves the following conjecture of Biernacki: If $P(\phi)$ is the area of a rectangle circumscribed about the planar oval R such that one of its sides makes an angle ϕ with a fixed direction l_0, then the following inequality holds:

$$\frac{S}{(2\pi)^{-1}\int_0^{2\pi} P(\phi)\,d\phi} \leqq \frac{\pi}{4},$$

where S is the area of the oval.

An application is made to prove two inequalities due respectively to Bieberbach and Hayashi.

P. C. Hammer (Madison, Wis.)

1950:

Besicovitch, A. S.; and Eggleston, H. G. The total length of the edges of a polyhedron. Quart. J. Math. Oxford Ser. (2) **8** (1957), 172–190.

Fejes Tóth [Lagerungen in der Ebene, auf der Kugel und im Raum, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953; MR **15**, 248; p. 143] proved that the edges of any convex polyhedron which contains a sphere of unit radius have a total length greater than 20. He conjectured that this length is greater than or equal to 24, with equality only when the polyhedron is a cube. The authors establish the truth of this conjecture. A crucial step in their argument (proved in an Appendix) is the fact that the function

$$\tan \tfrac{1}{8}(A+\pi)\{4\sin^2 \tfrac{1}{8}(A+\pi)-1\}^{\frac{1}{2}}/A$$

has a minimum where $A=1.42165\cdots$.

{Their methods would probably suffice to prove the somewhat similar conjecture that if the edges of a convex polyhedron form a "cage" surrounding a sphere of unit radius, then these edges have a total length of at least $9\sqrt{3}$.} *H. S. M. Coxeter* (Toronto, Ont.)

1951:

Mampel, K. L. Eine elementare Behandlung des Umdrehungsproblems von Kakeya. Math.-Phys. Semesterber. **5** (1957), 239–251.

The problem of Kakeya is the problem of moving a rigid straight-line segment AB in the plane in such a way as to sweep out a region of the least possible area while reversing the ends A and B. As is well known, under the additional condition that the region swept out must be convex, the solution region is the equilateral triangle Δ of altitude AB; but if this condition is removed then there is no solution region, the greatest lower bound of the areas of the regions in which reversal of ends is possible being 0. The author presents a detailed elementary construction for the latter result. The discussion proceeds in two steps: First, Δ is dissected and its parts reassembled in such a way as to yield regions $\Delta_1, \Delta_2, \cdots, \Delta_n, \cdots$ the respective areas of which form a null sequence; but in this construction the ability to reverse ends is lost. Next, the end-reversing property is regained through the exhibition of a sequence of related regions $\Delta_1', \Delta_2', \cdots, \Delta_n', \cdots$ having this property and at the same time retaining the property that their areas tend to 0 with increasing n.

E. F. Beckenbach (Los Angeles, Calif.)

1952:

Heppes, Aladár. On the partitioning of three-dimensional point-sets into sets of smaller diameter. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 413–416. (Hungarian)

Borsuk conjectured that every set in k-dimensional space of diameter 1 can be decomposed as the union of at most $k+1$ sets of diameter less than 1. For $k\leqq 2$ the conjecture is trivial. Recently Eggleston [J. London Math. Soc. **30** (1955), 11–24; MR **16**; 734] proved Borsuk's conjecture for $k=3$. The present paper gives a very much simpler proof of Borsuk's conjecture for $k=3$. For $k>3$ the conjecture remains open.

{A very similar proof has been found independently and almost simultaneously by B. Grunbaum [Proc. Cambridge Philos. Soc. **53** (1957), 776–778; MR **19**, 763].}

P. Erdös (Haifa)

1953:

Erdös, Pál; and Fejes Tóth, László. The distribution of points in a region. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 185–190. (Hungarian)

Let $P_1, \cdots, P_n$ be any n points of a given closed plane point set with area T, and let l_i be the distance from P_i of the nearest of the remaining $n-1$ points. If d_n is the maximum of $\min(l_1, \cdots, l_n)$ for varying choice of the P_i, it is known that $\lim_{n\to\infty} n^{\frac{1}{2}}d_n=(2T/\sqrt{3})^{\frac{1}{2}}$ [see L. Fejes Tóth, Lagerungen in der Ebene, Springer, Berlin-Göttingen-Heidelberg, 1953; MR **15**, 248]. This is now improved to $\lim_{n\to\infty} n^{-\frac{1}{2}}S_n=(2T/\sqrt{3})^{\frac{1}{2}}$, where S_n is the maximum of $l_1+\cdots+l_n$, for varying P_i. Related conjectures on shortest paths are mentioned.

F. V. Atkinson (Canberra)

1954:

Reeve, J. E. On the volume of lattice polyhedra. Proc. London Math. Soc. (3) **7** (1957), 378–395.

In the real affine plane, let $\bar{\gamma}$ be a simple polygon whose vertices all belong to a lattice l, and let γ be the closed region bounded by $\bar{\gamma}$. Let $l(\gamma)$, $l(\bar{\gamma})$ denote the number of lattice points belonging to γ, $\bar{\gamma}$, respectively. Steinhaus [Mathematical snapshots, Oxford, 1950, pp. 77–82; MR **12**, 44] proved that the area of γ, in terms of the fundamental parallelogram of l as unit, is $l(\gamma)-\frac{1}{2}l(\bar{\gamma})-1$. Although the obvious extension to three dimensions inevitably fails (since a tetrahedron having lattice points at its four vertices, but nowhere else, may have an arbitrarily large volume), the author has discovered and proved the following modified extension. In real affine 3-space, let $\bar{\Gamma}$ be the surface of a simply-connected polyhedral region Γ whose vertices all belong to the lattice L_1 consisting of the points whose affine coordinates are integers. Let L_n be the lattice of finer mesh consisting of the points whose coordinates are multiples of $1/n$. Let $L_n(\Gamma)$, $L_n(\bar{\Gamma})$ denote the number of points of L_n belonging to Γ, $\bar{\Gamma}$, respectively. Then $L_n(\bar{\Gamma})-2=n^2\{L_1(\bar{\Gamma})-2\}$; and the volume of Γ, in terms of the fundamental parallelepiped of L_1 as unit, is

$$\frac{L_n(\Gamma)}{n^3-n}-\frac{L_1(\Gamma)}{n^2-1}-\frac{L_1(\bar{\Gamma})}{2n+2}+\frac{1}{n}.$$

There are also some more general expressions covering non-simply-connected regions in terms of Euler-Poincaré characteristics. *H. S. M. Coxeter* (Toronto, Ont.)

1955:

Reeve, John-E. Seconde note sur le volume des polyèdres entiers. C. R. Acad. Sci. Paris **246** (1958), 2989–2991.

The author has simplified his statement of the results

in his earlier paper [reviewed above]. In real affine 3-space, let $V(\Gamma)$ denote the volume of a simply-connected polyhedral region Γ whose vertices all belong to the lattice L_1 consisting of the points whose affine coordinates are integers. Let L_n be the lattice of finer mesh consisting of the points whose coordinates are multiples of $1/n$. Let $L_n(\Gamma)$ denote the number of points of L_n belonging to Γ, counting each peripheral point by a suitable fraction (e.g., $\frac{1}{2}$ for a point inside a face). Then

$$(n-1)n(n+1)V(\Gamma)=L_n(\Gamma)-nL_1(\Gamma).$$

The first significant case is when $n=2$. The following 4-dimensional analogue is conjectured. Let $V(\Gamma)$ be the content of a polytope Γ whose vertices belong to a lattice L_1 in affine 4-space. Then

$$36V(\Gamma)=L_3(\Gamma)-3L_2(\Gamma)+3L_1(\Gamma).$$

H. S. M. Coxeter (Toronto, Ont.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 1706, 1726.

1956:

Viola, Tullio. Su un problema riguardante le congruenze degli insiemi di punti. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 290–293.

W. Sierpiński [Fund. Math. **37** (1950), 1–4; MR **12**, 597] proposed the problem of constructing a set E in the Euclidean plane, which is congruent to $E-\{p\}$ for every point $p \in E$. J. Mycielski [ibid. **42** (1955), 1–10; MR **17**, 244] answered the analogous problem affirmatively for the surface of the sphere, while E. G. Straus [Fund. Math. **44** (1957), 75–81] recently found that the answer for the original problem in the Euclidean plane Π is negative, i.e. in Π there exists at most one point $p \in E$ such that $E \cong E-\{p\}$. The author now intends to solve the analogous problem for the hyperbolic plane H, expecting to obtain a result similar to that for the surface of the sphere. In the present paper he constructs a suitable (denumerable) set E in H. In a second note he will further discuss this set E.

A. Rosenthal (Lafayette, Ind.)

1957:

Viola, Tullio. Su un problema riguardante le congruenze degli insiemi di punti. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 431–438.

This paper is the continuation of the first note with the same title [#1956 above]. In the present note the set E is further studied and, in particular, it is proved that E has the desired property, namely to be congruent to $E-\{p\}$ for every point $p \in E$. *A. Rosenthal* (Lafayette, Ind.)

1958:

Erdös, P.; et Marcus, S. Sur la décomposition de l'espace euclidien en ensembles homogènes. Acta Math. Acad. Sci. Hungar. **8** (1957), 443–452.

A non-null subset of euclidean n-space E_n is homogeneous (Borel) if for each two of its points X, Y the translation XY transforms the set in itself. The authors show that there does not exist any decomposition of E_n into a finite number k of pairwise mutually exclusive homogeneous sets for $k>1$. On the other hand, they prove that the E_n may be decomposed into m homogeneous, non-measurable sets E_t, superposable by translation, (and each containing a non-null perfect set) for $\aleph_0 \leq m \leq 2^{\aleph_0}$. Moreover, for each measurable set E of E_n, $\mu^*(E_t \cap E)=\mu(E)$, $\mu_*(E_t \cap E)=0$, where μ^*, μ, μ_*, denote exterior measure, measure, and interior measure, respectively. Similar results for the line have been established by Ruziewicz [Fund. Math. **5** (1924), 92–95] and Halperin [Proc. Amer. Math. Soc. **2** (1951), 221–224; MR **12**, 685].

L. M. Blumenthal (Columbia, Mo.)

1959:

Neugebauer, Christoph J. B-sets and fine-cyclic elements. Trans. Amer. Math. Soc. **88** (1958), 121–136.

Let P be a Peano space. A subset B of P is called a B-set if B is a nondegenerate continuum such that either B is P or the frontier of every component of $P-B$ is a finite set of points. Under the additional assumption that the degree of multicoherence $r(P)$ of P is finite it is established that the B-sets of P possess many properties analogous to those of the A-sets of P. A subset C of P is termed a fine-cyclic element of P if C is a B-set such that no finite set of points in C disconnects C. Under the additional assumption that $r(P)$ is finite it is shown that the fine-cyclic elements of P possess properties similar to the properties of the proper cyclic elements of P. If P is unicoherent then the fine-cyclic elements of P are identical with the proper cyclic elements of P. Let T be a continuous transformation from a closed finitely connected Jordan region J in the plane into three space, and let m be the monotone component of the monotone light factorization of T from J onto the middle space M. It is shown that a subset K of J is a fine-cyclic element of T in the sense of Cesari if and only if mK is a fine-cyclic element of M. *P. V. Reichelderfer* (Columbus, Ohio)

1960:

Grace, Edward E. Cut sets in totally nonaposyndetic continua. Proc. Amer. Math. Soc. **9** (1958), 98–104.

A totally nonaposyndetic, compact, nondegenerate, metric continuum contains at least one (weak) cutpoint [F. B. Jones, Amer. J. Math. **70** (1948), 403–413; MR **9**, 606; corollary to Theorem 12]. The author's generalization of this proposition (when narrowed down considerably) results in the following corollary: If (1) M is a perfectly separable, complete, metric continuum, (2) D is an open set such that M is nonaposyndetic at each point of D, and (3) Q is a countable subset of M, then D contains an inner limiting set [$=G_\delta$-set] I, dense in D, such that if x is in I and M is nonaposyndetic at x with respect to the point y, then y cuts x from each point of $(Q+y)-y$ in M. However, the main results are cut set existence theorems which hold true in rather general connected spaces called Baire topological (roughly, spaces in which a certain kind of category argument works) and yield a much closer association between nonaposyndesis, local nonaposyndesis, and (weak) cut sets. In particular, they are even applicable to certain finite connected spaces. Although the author's investigation is limited to the particular objectives indicated above, the class of abstract spaces that he defines are of interest in themselves. While it is clear that several kinds of spaces belong to this class, e.g., compact metric spaces, it is not at all evident that bicompact Hausdorff spaces do.

F. B. Jones (Chapel Hill, N.C.)

1961:

Burgess, C. E. Continua and various types of homogeneity. Trans. Amer. Math. Soc. **88** (1958), 366–374.

Suppose that M is a nondegenerate compact metric

continuum. When M has certain topological properties in addition to some sort of homogeneity, M is shown to be a simple closed curve. (M is n-homogeneous if for each pair of subsets H and K of n distinct points each, there is a homeomorphism T such that $T(M)=M$ and $T(H)=K$; M is nearly n-homogeneous if for each set H of n distinct points of M and each collection $\mathscr{C}$ of n open subsets of M there exist a set K of n distinct points (one from each element of $\mathscr{C}$) and a homeomorphism T such that $T(M)=M$ and $T(H)=K$. When $n=1$ we have the usual meaning of "homogeneous" and "nearly homogeneous.") These results extend and supplement known characterizations of a simple closed curve. Specifically: M is a simple closed curve provided that (1) M is homogeneous and (a) M is separated by some countable set or (b) M is hereditarily locally connected; or (2) M is 2-homogeneous and (a) M is multicoherent or (b) the homeomorphism may be chosen to leave any given subcontinuum fixed; or (3) M is nearly 2-homogeneous but not a triod; or (4) M is nearly 3-homogeneous, decomposable, and some two of its points are not separated in M by any subcontinuum of M.

Furthermore, when M is a subset of the plane which has a finite number (more than one) of complementary domains, then M is a simple closed curve provided that M is nearly homogeneous and (a) M contains a simple closed curve or (b) M is aposyndetic or (c) M contains no cut point or (d) M is arc-wise connected.

In this group of plane characterizations it is necessary to assume that M separates the plane, for there exists a dendron which is nearly homogeneous (its end points being dense in it); and it is also necessary to assume that M does not have infinitely many complementary domains, because of the well known example due to Sierpiński (the holey disk) which is nearly homogeneous.

Several rather interesting properties of a nearly homogeneous continuum M are obtained. One relates an indecomposable subcontinuum to its complement in M. Another states that if M lies in the plane but does not separate the plane, then M is either indecomposable or a triod.

Left partially unanswered is the question: Is every 2-homogeneous compact metric continuum locally connected? *F. B. Jones* (Chapel Hill, N.C.)

1962:

Tominaga, Akira. On pairs of domains in R^n with boundaries in common. J. Sci. Hiroshima Univ. Ser. A **21** (1957/58), 161–162.

Consider the following property that pairs of open sets D_1 and D_2 may have. (*) D_1 has the same boundary as D_2, and the interior of the closure of $D_1 \cup D_2$ is just $D_1 \cup D_2$. The author constructs in Euclidean space E^n two disjoint domains, each homeomorphic to E^n, each bounded in E^n, and such that the pair satisfies (*). {The hypothesis that the domains be bounded should, of course, be added to the statement of all the theorems in this paper.}

H. H. Corson (Seattle, Wash.)

1963:

Isiwata, Takesi. Some classes of completely regular T_1-spaces. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A **5** (1957), 287–292.

Let $B(X)$ denote the ring of all bounded continuous real-valued functions on a topological space X. The author observes that if X and Y are completely regular (T_1-spaces), and there is an isomorphism ϕ of $B(X)$ onto $B(Y)$ such that if $A \subset B(X)$, then $\phi[A]$ separates the points of Y if and only if A separates the points of X, then X and Y are homeomorphic. *M. Henriksen* (Lafayette, Ind.)

1964:

Isiwata, Takesi. On the ring of all bounded continuous functions. Sci. Rep. Tokyo Kyoika Daigaku. Sect. A **5** (1957), 293–294.

The following properties of a completely regular (T_1-) space X (all of which are implied by, and none of which imply, normality) are considered. (1) Any two disjoint closed subsets of X, one of which is countable, can be separated by a continuous real-valued function. (2) Any two disjoint closed subsets of X, one of which is countable, can be separated by disjoint open subsets. (3) Any continuous real-valued function defined on a countable closed subset of X has a continuous extension over X. (4) Any two disjoint countable closed subsets of X can be separated by a continuous real-valued function. (5) Any two disjoint countable closed subsets of X can be separated by disjoint open subsets.

It is shown that

$$(1)^* \Rightarrow (3) \Leftrightarrow (4)^* \Rightarrow (5) \Leftarrow^* (2) \Leftarrow (1),$$

where "*" indicates that the implication cannot be reversed. Moreover, (3) does not imply (2), but whether (2) implies (1) or (3) is left open. Countable paracompactness implies (2), and pseudo-compactness and (2) are together equivalent to countable compactness. X has property (i) if and only if every closed proper subspace has (i) ($i=1, \cdots, 5$). A product of two spaces with property (i) need not have (i) if $i=1, 2$, but for $i>2$ this is left open. *M. Henriksen* (Lafayette, Ind.)

1965:

Isiwata, Takesi. A generalization of Rudin's theorem for the homogeneity problem. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A **5** (1957), 300–303.

Let $\tilde{X}$ denote the Stone-Čech compactification of the completely regular (T_1-) space X. It is shown that if X admits an unbounded continuous real-valued function, and if the continuum hypothesis is true, then $\tilde{X}-X$ is not homogeneous. This conclusion was reached earlier by W. Rudin under the additional assumption that X is locally compact [Duke Math. J. **23** (1956), 409–419, 633; MR **18**, 324].

{The reviewer found the following misprints deceptive. On p. 301, l. −1, p. 302, l. 5, and p. 303, l. 3, change "X" (first occurrence) to "$\tilde{X}$"; p. 302, l. −2, replace "H_m" by "H_m contained in G_m"; p. 303, l. 10, change "f" to "g".}

M. Henriksen (Lafayette, Ind.)

1966:

Isiwata, Takesi. On subspaces of Čech compactification space. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. **5** (1957), 304–309.

A *-subspace of $\tilde{X}$ (defined as in the preceding review) is a proper subspace of $\tilde{X}$ that contains X properly. The main results of this paper are the following. If X is locally compact and normal, then every *-subspace of $\tilde{X}$ is normal if and only if $(\tilde{X}-X)$ is a finite set. If X is normal and not countably compact, then there is a non-normal *-subspace of $\tilde{X}$.

{Deceptive misprints: p. 307, l. 8, delete "and $\overline{\tilde{X}}^* < \aleph_0$"; p. 307, l. −13, change "X" to "X^*"; p. 308, l. 2, change "$\tilde{X}^*$" to "X^*".} *M. Henriksen* (Lafayette, Ind.)

1967:

Mrówka, S. Some properties of Q-spaces. Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 947–950, LXXX. (Russian summary)

Two theorems on Q-spaces (i.e. completely regular spaces X such that X is an intersection of F_σ-sets in the compactification βX) are proved: (1) if X is normal, $X=\cup A_n$, A_n are closed (in X) Q-spaces, then X is a Q-space; (2) if X is a Q-space, $A \subset X$ is an intersection of F_σ-sets (in X), then A is a Q-space.

M. Katětov (Prague)

1968:

Kerstan, Johannes. Eine Charakterisierung der vollständig regulären Räume. Math. Nachr. **17** (1958), 27–46.

The author gives characterizations of completely regular spaces, and other types of spaces such as normal, perfectly normal, etc., in terms of the lattice theoretic properties of a subbase. Translating back into the usual language of topology, the author establishes such standard results as: A regular space with a countable base is completely regular. The proofs seem to differ from those found in Bourbaki or Kelley [General topology, Van Nostrand, Toronto-New York-London, 1955; MR **16**, 1136] mostly in notation. However, if one wishes to introduce completely regular spaces without reference to the real numbers, the author gives a characterization in the standard terminology of topology which will serve that purpose, and which does not depend on the theory of uniform structures. *H. H. Corson* (Seattle, Wash.)

1969:

Ponomarev, V. I. On continuous decompositions of bicompacts. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 4(76), 335–340. (Russian)

Main results may be stated (in a terminology somewhat different from that of the author) as follows. Let $\{A\}$ be a decomposition (A closed) of a compact Hausdorff space R. Then (1) if $\{A\}$ is upper semi-continuous, then so is the decomposition consisting of all components of sets A; as a consequence, the monotone-light factorization theorem is valid for R (these results are well known for a compact metric R; as for the general case, it seems that they have never been stated explicitly, although perhaps virtually known); (2) $\{A\}$ is continuous if and only if the corresponding mapping is open [cf. G. Whyburn, Duke Math. J. **17** (1950), 69–74; MR **11**, 194; Th. 3]; (3) $\{A\}$ is continuous if and only if it is closed as a subset of the space of all closed non-void $F \subset R$.

M. Katětov (Prague)

1970:

Pervin, William J.; and Levine, Norman. Connected mappings of Hausdorff spaces. Proc. Amer. Math. Soc. **9** (1958), 488–496.

Suppose K and K^* are Hausdorff spaces and f is a single-valued connected map of K into K^* (i.e., fX is connected for each connected $X \subset K$). The authors prove that if K is locally connected and satisfies the first axiom of countability, then f is continuous at a point p of K if and only if $\lim f(p_n)$ exists in K for each sequence $p_n \to p$. If, in addition, K is compact and satisfies the first axiom of countability, then for each p in K the set $L(f; p)$ is connected, where $L(f; p)$ is the set of all $p^* \in K^*$ such that $p=\lim f(p_n)$ for some sequence $p_n \to p$. Their other theorems are mainly refinements of known results, especially those of Tanaka [J. Math. Soc. Japan **7** (1955), 389–393; MR **17**, 1116]. {To their list of general references on connected maps might be added Fan and Struble, Indag. Math. **16** (1954), 161–164 [MR **15**, 815], and F. B. Jones, Bull. Amer. Math. Soc. **48** (1942), 115–120 [MR **3**, 229].} *V. L. Klee, Jr.* (Copenhagen)

1971:

McDougle, Paul. A theorem on quasi-compact mappings. Proc. Amer. Math. Soc. **9** (1958), 474–477.

Let f be a quasicompact continuous (i.e., quotient) mapping of a locally compact separable metric space X onto a topological space Y. For Y also to be locally compact separable metric, it is necessary and sufficient that (1) f sends every compact set into a closed set, and (2) for each $y \in Y$ there is a compact $C \subset f^{-1}(y)$ such that, for every open $U \supset C$, $y \in \text{Int } f(U)$. Property (2) holds automatically if f is open, or if f is closed and each $f^{-1}(y)$ is compact (or, more generally, has a compact frontier). An example shows that (1) and (2) are not sufficient without the local compactness of X.

A. H. Stone (Manchester)

1972:

Tominaga, Akira. Note on fixed-point theorem. J. Sci. Hiroshima Univ. Ser. A **21** (1957/58), 7–14.

The author establishes a number of fixed point theorems for plane continua. Let M be a plane continuum which does not separate the plane. Assume that the interior of M has a finite number of components and that each component C has the "a.f.p.p." (meaning apparently that $\bar{C}$ has the fixed point property.) Then every map of M onto M has a fixed point. If $M=N_1 \cup N_2$ where N_1 and N_2 are arcwise components of M having the a.f.p.p. and $\bar{N}_1=\bar{N}_2$, then M has the fixed point property. There is also a fixed point theorem for multivalued maps of a simplex into itself. *P. A. Smith* (New York, N.Y.)

1973:

Bing, R. H. Necessary and sufficient conditions that a 3-manifold be S^3. Ann. of Math. (2) **68** (1958), 17–37.

According to a conjecture of Poincaré, any compact, connected 3-manifold is homeomorphic to S^3, the surface of a sphere in E^4, euclidean 4-space, provided it is simply connected. The author gives a partial solution to the problem posed by this conjecture by proving the following theorem. A compact, connected 3-manifold is homeomorphic to S^3 if each simple closed curve in the manifold lies in a topological cube in the manifold. The condition of the theorem that replaces simply connected in the conjecture is essentially a property suggested by R. H. Fox and E. Artin [Ann. of Math. (2) **49** (1948), 979–990; MR **10**, 317] as a likely crucial condition in characterizing S^3. A corollary to the proof of this theorem asserts that a bounded connected open subset U of E^3 is topologically E^3 provided the boundary of U is connected and each polyhedral simple closed curve in U lies in a topological cube in U. Theorems are also established using simple connectivity. For example, the author exhibits a cube, M_1, with a knotted hole in it as the sum of a cube with two handles and a topological cube and proves that if a 3-manifold, M, is the sum of M_1 and a torus, M_2, such that $M_1 M_2$=boundary M_1=boundary M_2, then M is topologically S^3 provided it is simply connected.

W. R. Utz (Columbia, Mo.)

ALGEBRAIC TOPOLOGY

See also 1678, 1973.

1974:

Suzuki, Haruo. On the realization of homology classes by submanifolds. Trans. Amer. Math. Soc. **87** (1958), 541–550.

The main theorem of this paper may be stated as follows: Let u be an n-dimensional cohomology class (mod 2) in a compact differentiable manifold of dimension $m+n$. If u is realizable for the orthogonal group $O(k)$ in the sense of Thom [Comment. Math. Helv. **28** (1954), 17–86; MR **15**, 890] with $k \leq n$, then the Steenrod square of u, $Sq^k(u)$, is also realizable. The author also proves an assertion of Thom [loc. cit.] to the effect that cup-products of realizable cohomology classes are also realizable.

W. S. Massey (Providence, R.I.)

1975:

Suzuki, Haruo. On the realization of the Stiefel-Whitney characteristic classes by submanifolds. Tôhoku Math. J. (2) **10** (1958), 91–115.

Let V_n be a compact differentiable manifold of dimension n and let $\mathscr{E}^m$ be an m-vector bundle over V_n; the author asks if the Stiefel-Whitney classes $W_i(\mathscr{E}^m)$ of the bundle $\mathscr{E}^m$ are realizable by submanifolds of V_n. In chapter I the problem is approached directly. Let f: $V_n \to G_{m,N}$ be a map of V_n into the Grassmannian $G_{m,N}$ inducing $\mathscr{E}^m$; then, by a theorem of Thom, f is homotopic to a map f_1 which is transversally regular with respect to singular subvarieties of the Schubert variety F_i representing W_i in $G_{m,N}$. The author calls $f_1^{-1}(F_i)$ the induced Schubert variety (induced Schubert manifold if it is without singularities); it has a singular subvariety S_1 which is a realization of the f^*-image of the Schubert class $\{0, \cdots, 0, 2, \cdots, 2\}$ (2 appears $i+1$ times). Put $f^*\{0, \cdots, 0, 2, \cdots, 2\} = S_1^*$. Then S_1^* vanishes if $f_1^{-1}(F_i)$ is a manifold and may be regarded as the first obstruction to realizing $W_i(\mathscr{E}^m)$ by an induced Schubert manifold. Since $\{0, \cdots, 0, 2, 2, 2\} = W_2W_4 + W_3^2$ the author obtains Theorem 2.1: If $W_2(\mathscr{E}^m)$ is realizable by an induced Schubert manifold, then $W_2(\mathscr{E}^m)W_4(\mathscr{E}^m) = (W_3(\mathscr{E}^m))^2$. Using Wu's relations among the W_i, the author proves Theorem 2.2: If $W_{2j+1}(\mathscr{E}^m)$ is realizable by an induced Schubert manifold, then the Pontryagin class $P_{4(j+1)}(\mathscr{E}^m)$ is zero and $W_{2j+1}(\mathscr{E}^m)W_{2j+3}(\mathscr{E}^m) = (W_{2(j+1)}(\mathscr{E}^m))^2$. After giving examples (which serve also to show that the property of realizability by induced Schubert manifolds is not invariant under cobordism) the author ends the chapter by giving sufficient conditions for realizability. Thus $W_i(\mathscr{E}^m)$ can be realized by an induced Schubert manifold if $n < 2(i+1)$ and $W_{2k+1}(\mathscr{E}^m)$ can be realized by an induced Schubert manifold if $n = 4(k+1)$ and the Pontryagin class $P_{4(k+1)}(\mathscr{E}^m)$ vanishes.

In Chapter 2 the main tool is Thom's fundamental realizability theorem [Comment. Math. Helv. **28** (1954), 17–86; MR **15**, 890]. Using this and Wu's relations, the author proves that if $W_{2i}(\mathscr{E})$ is realizable by a submanifold then $W_{2i+1}(\mathscr{E})^2$ belongs to the ideal generated by $W_{2i}(\mathscr{E})$; a similar but more complicated condition is given for the realizability of $W_{2i+1}(\mathscr{E})$. The condition on $W_{2i}(\mathscr{E})$ is used to show that if $\mathscr{E}$ is the tangent bundle to $P = P(2) \times P(4) \times P(5)$ then $W_2(\mathscr{E}) = W_2(P)$ is not realizable by a submanifold. Detailed computations of the early terms of the Postnikov system for $M(O(2))$ enable the author to prove that in an orientable 6-manifold V_6 the W_2 class of any vector bundle is realizable. No example is known of a non-orientable 6-manifold with a W_2 which is not realizable; for the generators $P(6)$, $P(4) \times P(2)$, $P(2) \times P(2) \times P(2)$ of the mod 2 cobordism group in dimension 6 the sufficient condition for realizability given by the author (Theorem 7.1) is satisfied.

In Chapter 3 the author proves that the Stiefel-Whitney classes of the normal bundle over a complete intersection V of non-singular hypersurfaces in a compact differentiable manifold M are realizable by submanifolds of V. The Stiefel-Whitney classes of the tangent bundle over V are also realizable if the mod 2 cohomology of M is realizable.

{Misprints appear as relentlessly as we have come to expect. Particularly troublesome is the statement of Theorem 5.1, where "W_i" should be "W_{2i}", and the proof: we should have "$\cdots = f^*(U_{2i})f^*(U_{2i}W_1^2) = W_{2i}(\mathscr{E})f^*\phi_{G,2i}^*(W_1^2) = \cdots$". Also (p. 96) in the statement of Theorem 2.2 "$W_{4(j+1)}$" should be "$W_{2(j+1)}$".}

P. J. Hilton (Ithaca, N.Y.)

1976:

Simon, Arthur B. n-cyclic elements: correction. Duke Math. J. **24** (1957), 681.

A counterexample (by E. Dyer) is given showing that Theorem 4.9 (concerning inequality $\operatorname{ind} X \leq \operatorname{cd} X + 2$) in the author's paper [Duke Math. J. **24** (1957), 1–7; MR **19**, 302] is not valid.

M. Katětov (Prague)

1977:

Cohen, Haskell. A correction. Duke Math. J. **25** (1958), 601.

The paper corrected is the author's [same J. **21** (1954), 209–224; MR **16**, 609].

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 1832, 1838, 1856, 1975, 2101, 2222.

1978:

***Kreyszig, Erwin. Differentialgeometrie.** Mathematik und ihre Anwendungen in Physik und Technik, Reihe A, Bd. 25. Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1957. xi+421 pp. DM 36.00.

This book is volume 25 of the well-known collection "Hilb". The author points out the extension and the character of "general mathematical knowledge" of differential geometry, and he promises, in the introduction of his book, that this will become obvious in his description. A similar program has been taken as basis by W. Blaschke and his coworkers in his well-known "Vorlesungen über Differentialgeometrie" [Bd. I, 4. Aufl., 1945, Bd. II, 1923, Bd. III, 1929, Springer, Berlin]. At that time it was group theory — above all the systematic extension of F. Klein's "Erlanger Programm" to differential geometry — that disclosed the wide range of the subject and the prospect of the course to be adopted. The relationship with methods of the variation calculus has, of course, not been neglected. To avoid repetitions recent authors of compendiums on differential geometry have done without such a general theoretical outlook. But the author, a faithful follower of the famous school of function theory established by Professor Behnke at the German university of Münster, succeeds in attending to "general knowledge" in the borderland of function theory and differential geometry.

After an introductory chapter, which is chiefly reserved to vector calculus in Euclidean three-dimensional space, chapter two deals, in more than sixty pages, with a detailed

theory of curves in the Euclidean three-dimensional space. From the first the author strives to illustrate his book by very instructive and well-shaped figures (105 for 366 pages of text). The reader will find geometry as well as calculus, but of the latter only as much as is required to accomplish the geometrical program. In the theory of surfaces, pieces of surfaces are represented as topological images corresponding to the pieces of a parameter plane. By the end of the third chapter, the author has worked out the concept of the Riemannian space. Tensor calculus in the Riemannian space is developed on the model of H. Weyl: Scalars are tensors of degree zero, vectors are tensors of the first degree. Tensor algebra is dealt with in little more than thirty pages, but that is quite sufficient for the geometrical program of this chapter. The concept of a tensor is grasped through its usual transformation law. To put up a front of "precision" by a coordinateless definition of tensors might have been avoided, since the reader who is interested in differential geometry will certainly have overcome his fear of indices and coordinates. {Bold-face type of the "convention of summation" — which is a well known rule in tensor calculus — struck the reviewer as an exaggeration, since the text of the incomparably more important "theorema egregium" does not appear in heavy type.} Among the remaining geometrical topics of this chapter, H. A. Schwarz's remarks on the definition of the area of a piece of a surface are worth mentioning. The embedding theory of surfaces in the Euclidean three-dimensional space is given in the fourth chapter, especially in its intuitive part, i.e., the theory of principal curvatures, of the Dupin-indicatrix and related things such as flat points, saddle points, umbilical points. The text contains the equations of Weingarten and Gauss, the theory of Christoffel symbols and the curvature tensors of Riemann and Christoffel (of course that theory is restricted to the binary domain). In connection with the theory of Christoffel symbols, the existence of geodesic coordinates can be proved, but the theory of geodesic lines and geodesic curvature is reserved for the fifth chapter. The most important subject in the fifth chapter is the "theorema egregium" of Gauss-Bonnet and its applications to the theory of closed surfaces of genre p. The theory of developable surfaces is developed in the sixth and eighth chapter. The sixth chapter contains a detailed theory of representations of surfaces as images of other surfaces. In Euclidean three-dimensional embedding spaces the following theorem holds: a (sufficiently small) piece of a surface of the class $r \geqq 3$ can be isometrically represented in a plane if and only if this piece is a piece of a developable surface. (On the other hand, the theory of developable surfaces can be combined with the theory of curves as suggested by the principle of duality.) In these paragraphs, the theory of functions becomes more and more important. In particular, one may mention § 64 concerning the Bergmann metric, which is an invariant one by conformal representations. Its "kernel-function" is given by

$$Q_B(z, \bar{t}) = \sum_{p=1}^{\infty} \Phi_p(z)\Phi_p(\bar{t})$$

(B domain in the z-plane). The functions $\Phi_p(z)$ are elements of a complete orthonormal system and the metrical fundamental form is given by

$$ds_B^2 = Q_B(z, \bar{z})|dz|^2 = Q_B(z, \bar{z})(dx^2+dy^2).$$

In the following, the author deals with the stereographic representation and the representations of Mercator, Lambert, Sansone and Bonnet. An important chapter is the seventh: absolute differentiation and parallel displacement. The corresponding tensor calculus leads to the identities of Ricci and Bianchi and to the differential parameters of Beltrami. The parallelism of Levi-Cività is showed to be equivalent to the postulate: the displacement of the vector a^α along the geodesic line C should be carried out in such a way that the angle between the displaced vector and the tangent vector of C is a constant angle for all points of C. Then the following differential equations hold:

$$\frac{da^\alpha}{ds} + \Gamma_{\gamma\sigma}^{\alpha} a^\gamma \dot{u}^\sigma = 0,$$

which now can be used as a definition of parallel displacement even if C is not a geodesic line. In the same chapter the author deals with some generalizations of the absolute derivatives and parallelisms (torsion). The last chapter concerns special surfaces, especially minimal surfaces and the "Betragsflächen" of E. Ullrich and its instructive figures (regarding the functions $\sin \pi z$ and $(\sin \pi z)^{-1}$). Finally, the author deals with families of surfaces depending on one parameter; tubes, central surfaces, parallel surfaces, surfaces of constant Gaussian curvature, surfaces of rotation (spherical and pseudospherical), geodesic representation and relations to the noneuclidean Geometry. To every section of the book some exercises are attached, whose solutions are given separately. The bibliographical and historical remarks given by the author increase the attraction of this beautiful book.

M. Pinl (Cologne)

1979:

Godeaux, Lucien. Sur la détermination des courbes tracées sur une surface multiple. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 356–363.

1980:

Riftin, L. P. General questions of geometry on the surfaces of conoids. Trudy Inst. Mašinoved. Sem. Teorii Mašin i Mehanizmov **16** (1956), no. 63, 41–63. (Russian)

Verfasser entwickelt in dieser Untersuchung eine explizite Berechnung der differentialgeometrisch wichtigen Krümmungsgrößen, insbesondere der Gaußschen Krümmung K und der mittleren Krümmung H für verschiedene Klassen von Konoidflächen allgemeiner und verallgemeinerter Art, deren weitere Unterteilung vom Verhalten der Parameter ε und δ in der Vektorgleichung

$$\mathbf{r} = (\eta \cos \Phi - \varepsilon \sin \Phi)\mathbf{i} - (\eta \sin \Phi + \varepsilon \cos \Phi)\mathbf{j} - \frac{z}{\cos \delta}\mathbf{k}$$

der verallgemeinerten Konoidflächen abhängt ($\mathbf{r}$=Ortsvektor, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ Einheitsvektoren in den Achsen eines kartesischen Koordinatensystems, Φ, z Gaußsche Flächenparameter, $\eta = s_0 + s(\Phi, z)$). Bedeutet K die Gaußsche Krümmung eines verallgemeinerten exzentrischen Konoids und K^* die Gaußsche Krümmung eines ebensolchen allgemeinen Typus ($\delta = 0$), so gilt $K < K^*$. Im zentrischen Fall ($\varepsilon = 0$) führt die analoge Untersuchung auf die Gaußschen Krümmungen $K' < K^{*\prime}$ (K' entspricht $\varepsilon = 0$, $K^{*\prime}$ entspricht $\varepsilon = \delta = 0$). In weiteren drei Abschnitten behandelt Verfasser kinematisch-technisch wichtige Konoidflächen, wie sie sich insbesondere aus der Konstruktion schwingender Hebel ergeben, deren Schwingungen von den exzentrischen Drehungen der Konoide abgeleitet werden (die Kontaktspitze des Hebels gleitet auf der

Oberfläche des rotierenden Konoids). Dabei werden als wichtige Spezialfälle (unter anderen) der Fall paralleler bzw. orthogonal kreuzender Konoid- und Hebeldrehachsen durchgerechnet. — In einem zweiten Teil untersucht Verfasser die Möglichkeiten, die vorher gewonnenen rein flächentheoretisch abgeleiteten Ergebnisse praktischen Anforderungen der Technik anzupassen. Dabei kommt es auf gewisse Abrundungen der theoretisch gewonnenen Flächen an, für welche Einhüllungsprozesse durchgeführt mit zweiparametrigen Flächenscharen $F(x, y, z, C_1, C_2)=0$ durch Differentiation nach den Parametern C_1 und C_2 und Übergang zur einhüllenden Eliminantenfläche $\Phi(x, y, z)=0$ die mathematische Behandlungsmethode anzeigen. *M. Pinl* (Köln)

1981:

Babiĭ, V. P. On the volume of a layer covering a given surface. Uman. Ped. Inst. Nauk. Zap. 3 (1956), 89–92. (Ukrainian)

On the positive normal to a given closed surface Γ, with $ds^2=Edu^2+2Fdudv+Gdv^2$, a segment of length r is set out. The volume of the layer thus formed is $Sr+Mr^2+\frac{1}{3}Ir^3$, where S is the surface area of Γ, M the surface integral of the mean curvature, and I that of the Gaussian curvature. *D. J. Struik* (Cambridge, Mass.)

1982:

Artobolevskiĭ, I. I. The theory of mechanisms for drawing pedal curves and its application to cyclic curves. Trudy Inst. Mašinoved. Sem. Teorii Mašin i Mehanismov 17 (1957), no. 65, 37–72. (Russian)

1983:

Bakelman, I. J. Generalized solutions of Monge-Ampère equations. Dokl. Akad. Nauk SSSR (N.S.) 114 (1957), 1143–1145. (Russian)

Let Φ be a convex surface, $z=f(x, y)$ its equation, D a plane domain. For every $(x_0, y_0) \in D$ there is a support plane $z-f(x_0, y_0)=p_0(x-x_0)+q_0(y-y_0)$, and the correspondence $(x_0, y_0)\to(p_0, q_0)$ defines a mapping ψ from D to (p, q) plane. Let $R(p, q)$ be continuous function for all p, q and $R(p, q)\geqq R_0=\text{const}>0$. The author calls ψ the normal mapping of Φ, and $\omega_R(M)=\iint_{\psi(M)} 1/R(p, q)\,dpdq$ the R-area of the normal mapping of the set MCD. This generalizes the area of the spherical image of Φ (when $R=(1+p^2+q^2)^{3/2}$). If in the Monge-Ampère equation $rt-s^2=\varphi(x, y)R(p, q)$, $\varphi(x, y)>0$ in D and $R(p, q)$ as above, then every twice differentiable solution in D represents a convex surface Φ for which the normal mapping ψ is one-to-one, and

$$(*) \qquad \iint_M \varphi(x, y)dxdy=\iint_{\psi(M)} \frac{dpdq}{R(p, q)}=\omega_R(M)$$

for every Borel set MCD. The author calls a function $z=f(x, y)$ a generalized solution of the M-A equation if it satisfies (*), and proceeds to find conditions for existence of a convex surface Φ with a given R-area, imposing suitable boundary conditions. The results are extensions of A. D. Aleksandrov's work [Intrinsic geometry of convex surfaces, OGIZ, Moscow-Leningrad, 1948; Convex polyhedra, Gostehizdat, Moscow-Leningrad, 1950; MR 10, 619; 17, 74; 12, 732] for convex polyhedra or for special cases when R-area is the area of the spherical image.

V. Linis (Ottawa, Ont.)

1984:

Petrenko, A. I. Interpretation of coordinates and projections of Gauss, Mercator and Soldner on a sphere as perspective-conic coordinates and projections. Ukrain. Mat. Ž. 10 (1958), no. 1, 78–82. (Russian)

1985:

Fréchet, Maurice. Détermination des surfaces minima du type $a(x)+b(y)=c(z)$. II. Quadratures. Rend. Circ. Mat. Palermo (2) 6 (1957), 5–32.

Der Verf. führt die Beweise zu der im ersten Teil [C. R. Acad. Sci. Paris 244 (1957), 145–147; MR 19, 307] gegebenen Aufstellung der im Titel genannten Flächen zum Abschluß. Nachdem im ersten Teil der Arbeit die Funktionen $\alpha(u)=A'(u)^{-2}$, $\beta(v)=B'(v)^{-2}$ und $\gamma(w)=C'(w)^{-2}$ durch Lösen einer Funktionalgleichung bestimmt worden waren, ergeben sich A, B und C und damit die Parameterdarstellungen der gesuchten Flächen durch Quadraturen. Schließlich werden die Gleichungen in rechtwinkligen Koordinaten durch Eliminationen gewonnen; sie enthalten nur elementare und elliptische Funktionen. Es gibt 0-, 1- und 2-parametrige Scharen solcher Flächen.

K. Krickeberg (Heidelberg)

1986:

Marcus, F. Surfaces minima projectives qui sont simultanément minima euclidiennes. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. 8 (1957), no. 1, 19–36. (Romanian. Russian and French summaries)

It is shown that a minimal Euclidean surface which has a family of lines of curvature consisting of geodesics is minimal affine if it is minimal geodesic, and conversely. This suggests looking for the surfaces which are minimal in the euclidean, affine, and projective senses. The equations of these surfaces are found in finite form. The surfaces are imaginary and have first fundamental form $ds^2=(Au+Bv+C)^2(du^2+dv^2)$, with constants A, B, C such that $A^2+B^2+1=0$, when referred to their asymptotic curves. These surfaces belong to a class described by Tziteica-Wilczynski; they are also of the third type of Terracini; they are also such that their Euclidean normals are principal lines of Fubini.

The author has defined I-congruences [same Stud. 3 (1952)] formed by normals to minimal surfaces and having minimal surfaces for focal nappes. The mean surface of an I-congruence is a minimal surface in the euclidean, affine, and projective sense, and conversely.

A. Schwartz (New York, N.Y.)

1987:

Backes, F. Sur une question relative aux systèmes triples orthogonaux. Acad. Roy. Belg. Cl. Sci. (5) 43 (1957), 63–65.

L'A. dimostra che associando alle funzioni β_{ik}, introdotte da Darboux nello studio dei sistemi tripli ortogonali, altre sei convenienti funzioni, si ottiene un sistema che ammette soluzione in virtù di un teorema di esistenza di Darboux. *L. Amerio* (Milano)

1988:

Mihăilescu, Tiberiu. Réseaux conjugués à congruences associées paraboliques. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. Fiz. 7 (1956), no. 1–4, 15–49. (Romanian. Russian and French summaries)

It is known that a conjugate net in ordinary projective space P_3 has two associate congruences, namely, the axis and ray congruences. The purpose of this paper is to determine conjugate nets in P_3 with one or two parabolic associate congruences. On an arbitrary surface in P_3 there exist infinitely many conjugate nets such that their curves are not plane curves or conics and each net has one parabolic associate congruence. These conjugate nets depend on two arbitrary functions of one variable, and the asymptotic curves of the unique focal surface of

the parabolic associate congruence of each net do not correspond to the curves of the net.

The conjugate nets in P_3 with two parabolic associate congruences depend on six arbitrary functions of one variable, and their Laplace transforms are in asymptotic correspondence. Furthermore, the unique focal surface of the ray congruence of each one of such nets is one of the two focal surfaces of the congruence generated by the line through the unique focus of the ray as well as the axis at each point of the net.

C. C. Hsiung (Bethlehem, Pa.)

1989:

Kovancov, N. I. One-parameter families of congruences with ruled focal surfaces. Mat. Sb. N. S. **42(84)** (1957), 45–64. (Russian)

In this paper there are studied one-parameter families of congruences with ruled focal surfaces, which the author calls complexes C. They are a generalization of complexes K_3 introduced by the same author [Mat. Sb. N.S. **41(83)** (1957), 333–360; MR **19**, 450]. Employing the method of Cartan's repère mobile, it is shown that they depend on six functions of two variables. The geometrical construction of complexes C may be described as follows. If a pair of congruences S and Σ is stratified into one parameter families of ruled surfaces and if an arbitrary one-to-one correspondence between the surface s of the one family and the surface σ of the other is established, then the one-parameter family of congruences (s, σ) with focal surfaces s, σ generates complex C.

The class of complexes C with developable surfaces s and σ ("torse complexes") depends only on four functions of two variables. The following construction of torse complexes is established: On the arbitrary surface F $[G]$ let there be given a one-parameter family of curves f $[g]$. If an arbitrary one-to-one correspondence between the curves f and g is established, then the intersections of tangent planes of surfaces F and G at the points of corresponding curves f and g generate a torse complex.

The remainder of the paper gives results on complexes C, for which (s, σ) are W-congruences; they arise when the generators of congruences S and Σ are in a one-to-one correspondence. Special results are given for the case when congruences S and Σ represents a pair K (i.e. when the focal points of one congruence coincide with the focal plane of the other). *A. Urban* (Prague)

1990:

Mishra, R. S. Congruences of curves in a subspace of a non-Riemannian space. Riv. Mat. Univ. Parma **7** (1956), 235–241.

La considération d'un système de $m-n$ congruences de courbes d'un espace non-Riemannien, permet à l'auteur d'établir simplement la généralisation aux espaces non-Riemanniens des équations généralisées de Gauss et de Riemann, de même que les généralisations que Voss et Ricci ont données des équations de Gauss et de Mainardi-Codazzi pour les sous-espaces plongés dans un espace non-Riemannien. *P. Vincensini* (Marseille)

1991:

Marcus, F. Propriétés métriques des surfaces $\mathscr{E}$ de Cartan. Bul. Inst. Politehn. Iaşi (N.S.) **2** (1956), no. 3-4, 27–33. (Romanian. Russian and French summaries)

An E-surface of Cartan in projective 3-space is one which admits an E-network; $h=\bar{h}$ and $k=\bar{k}$, h, k, $\bar{h}$, $\bar{k}$ being the Darboux invariants for the Laplace equations for the network. Here the author works from a metric point of view and looks for surfaces whose lines of curvature are networks for which $h=\bar{h}$ and $k=\bar{k}$. If the fundamental forms of a surface referred to its lines of curvature are Edu^2+Gdv^2 and edu^2+gdv^2, the surfaces desired are characterized by $\partial^2 \log(e/E)/\partial u\partial v=\partial^2 \log(g/G)/\partial u\partial v=0$.

The author investigates surfaces whose lines of curvature have given spherical images. There are E-surfaces among these with metrics of the form $ds^2=du^2+udv^2$ and $ds^2=vdu^2+dv^2$ (surfaces of Weingarten).

A. Schwartz (New York, N.Y.)

1992:

Marcus, F. Compléments à quelques résultats de géométrie projective differentielle. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **8** (1957), no. 1, 37–47. (Romanian. Russian and French summaries)

Results of A. Pantazi, S. Finikoff, and T. Mihailescu on networks of curves on a surface are tied together and advanced, using methods which do not leave the surface. Let h, k, $\bar{h}$, $\bar{k}$ be the invariants of the Laplace equations for a network. (1) $h=\bar{h}$ and $k=\bar{k}$ for a conjugate network if and only if the two families of curves are parageodesics of Fubini. (2) $h=-\bar{h}$ and $k=-\bar{k}$ for a network if and only if the two families of curves are projective lines of Segre. (3) $h/\bar{h}=k/\bar{k}$ for a network if and only if the Bompiani invariants for the curves of one family equal those for the curves of the other family.

Finikoff showed that if two consecutive networks of a Laplace sequence were networks of Slotnick, all were, and conjectured that all the surfaces of the sequence were quadrics. The present author gives an example of a Laplace sequence of Cartan networks which do not belong to quadrics, this forcing a modification of Finikoff's conjecture. *A. Schwartz* (New York, N.Y.)

1993:

Marcus, F. Sur les réseaux de Koenigs. Rev. Math. Pures Appl. **2** (1957), 555–559.

Let h, k, $\bar{h}$, $\bar{k}$ be the Laplace invariants corresponding to an isothermal-conjugate network on a surface. The author shows that if $k=\bar{k}=0$, then $h=\bar{h}=0$ also. This can be restated: If the curves of one family of an isothermal-conjugate network are plane and conics, then the curves of the other family are also. The network is a Koenigs network and the curves lie in two pencils of planes.

A. Schwartz (New York, N.Y.)

1994:

Stein, E. A theorem on generalised Segre varieties. Rend. Mat. e Appl. (5) **16** (1957), 281–285.

L'auteur applique une méthode qu'il a utilisée dans deux publications antérieures [E. Stein, Nederl. Akad. Wetensch. Proc. Ser. A. **59** (1956), 104–107; Bull. Soc. Roy. Sci. Liège **26** (1957), 19–28; MR **17**, 1131; **19**, 318], à la démonstration d'un théorème généralisant le résultat bien connu de C. Segre suivant lequel une collinéation entre deux plans est représentée sur leur produit par une surface de Véronèse. Les notations étant celles des articles cités, ce théorème consiste en ce que toute paracollinéation entre une $V_n^{(h)}$ et une $V_n^{(k)}$ appartenant à un même espace à n dimensions est représentée, sur leur produit, par une $V_n^{(h+k)}$. *P. Vincensini* (Marseille)

1995:

Švec, Alois. Certaines enveloppes des familles ∞^3 d'homographies dans S_5. Czechoslovak Math. J. **7(82)** (1957), 57–65. (Russian summary)

Verfasser nennt Ebenenkomplexe des projectiven S_5 die von drei Parametern abhängenden Ebenenmengen,

wovon also im allgemeinen durch jeden Punkt des S_5 Ebenen der Menge gehen. Wie bei den dazu analogen Geradenkongruenzen des S_3 nennt man Brennpunkte einer Komplexebene ihre Schnittpunkte mit Nachbarebenen. Developpable V_3 aus ∞^1 Ebenen des Komplexes erzeugt, werden dann analog erklärt wie die Developpablen einer Kongruenz des S_3. Verf. beschränkt sich weiterhin auf solche Komplexe, bei denen die Brennpunkte in jeder Ebene drei linear unabhängige Geraden bilden. Hierzu bietet sich von selber in jeder Ebene ein Koordinatendreieck an, und man beschreibt den Komplex durch Pfaffsche Formen, Ableitungsgleichungen usw. Die Komplexe mit der genannten Beschränkung hängen von sechs Funktionen zweier Variablen ab. Man sagt zwei derartige Komplexe K und K' sind in der Korrespondenz C, wenn ihre Ebenen, aber auch die developpablen Teilmengen von je ∞^1 Ebenen einander zugeordnet sind. Nach E. Čech spricht man dann davon, daß K und K' punktdeformierte voneinander sind, wenn eine derartige Korrespondenz, die nur zwischen den Ebenen erklärt ist, sich auch auf die Punkte so ausdehnen läßt, dass entsprechende Ebenen projektiv aufeinander bezogen sind. Es wird dann gezeigt, daß alle aus einem gegebenen Komplex durch eine solche Punktdeformation hervorgehenden Komplexe von 18 Funktionen einer Variablen abhängen. *W. Burau* (Hamburg)

1996:

Švec, Alois. Remarques sur la théorie des déformations des congruences de droites. Czechoslovak Math. J. 7(82) (1957), 66–72. (Russian summary)

Gegeben seien die beiden projektiven Räume S_3 und S_3' sowie die Geradenkongruenzen LCS_3 und $L'CS_3'$. Zwischen S_3 und S_3' werde eine Punktabbildung C definiert, welche die Geraden von L in die von L' überführt und auf den Geraden je eine Projektivität ist, die auch die Fokalpunkte von L und L' einander zuordnet. Solche Abbildungen heißen punktweise developpable Korrespondenzen. Analog kann man dazu dual auch planar developpable Korrespondenzen erklären. Unter Benutzung Čechscher Begriffsbildungen wird die punktweise developpable Korrespondenz C noch so spezialisiert, daß man von C verlangt, es solle in jedem Punkt $X'=C(X)$ eine berührende Kollineation K existieren, derart daß der Kongruenzstrahl durch Y die K-linearisierende Gerade ist für alle Geraden durch Y in den Fokalebenen dieses Strahles. Für solche Korrespondenzen wird dann gezeigt, daß sie auf folgende Weise erzeugt werden: Man nehme zwei nichtparabolische Kongruenzen L und L', die durch planare projektive Verbiegung zusammenhängen. Dann nehme man die Gesamtheit der projektiven Abbildungen der Geraden von L auf die von L', die durch diese Beziehung bedingt sind. *W. Burau* (Hamburg)

1997:

Švec, Alois. Congruences de droites dans les espaces réglés à connexion projective. Czechoslovak Math. J. 7(82) (1957), 96–114. (Russian summary)

Unter Liniengeometrie in einem Raum mit projektiven Zusammenhang versteht der Verf. folgende Verallgemeinerung der Liniengeometrie des projektiven S_3: Den Punkten eines vierdimensionalen Bereichs M werden projektive Räume S_3 zugeordnet, wobei in jedem derselben eine feste Gerade liegt. Zwischen den verschiedenen lokalen S_3 bestehen alsdann projektive Abbildungen, die in üblicher Weise stetig vom Wege abhängen. Zur analytischen Behandlung wird in jedem lokalen S_3 ein Simplex A_1, A_2, A_3, A_4 angenommen, derart daß A_1A_2 die ausgezeichnete Gerade ist. Der projektive Zusammenhang zwischen den S_3 wird dann nach Cartanschem Muster durch Formeln $dA_i=\omega_{ij}A_j$ mit der Integrabilitätsbedingung

$$d\omega_{ij}=[\omega_{ik}\omega_{kj}]+R_{ij}{}^{\alpha\mu\beta\nu}[\omega_{\alpha\mu}\omega_{\beta\nu}] \quad (\alpha,\beta=1,2;\ \mu,\nu=3,4).$$

Man hat so eine Menge von ∞^4 Geraden, auf die sich die Liniengeometrie des gewöhnlichen S_3 weitgehend übertragen läßt. In der vorliegenden Arbeit geschieht das für die Theorie der Regelscharen und Kongruenzen. Es werden folgende Begriffe übertragen: Torse, Brennpunkt, wovon jede Kongruenz auch im allgemeinen zwei hat, wenn sie nicht parabolisch ist mit einem Brennpunkt oder planar mit unendlich vielen. Durch duale Auffassung der begleitenden S_3 und darin liegenden Geraden kommt man zu einem analogen Formelapparat, und jede Kongruenz kann somit wie im gewöhnlichen Fall gleichzeitig dual aufgefaßt werden. Weiterhin werden drei Invarianten der Kongruenz zugeordnet: I, I_1 und I_2. Davon ist J das Analogon der Invariante von Waelsch, I_1 und I_2 sind hier neu zu betrachten. Das Verschwinden von I_1 oder I_2 bedeutet, daß die Developpablen der Kongruenz auf der einen oder anderen Brennfläche Kurvennetze ausschneiden, die harmonisch konjugiert zu dem Asymptotennetz sind. Es werden dann noch folgende Beziehungen zwischen zwei verschiedenen Kongruenzen betrachtet: Torsale Abbildung und Deformation 2. Ordnung.

W. Burau (Hamburg)

1998:

Grunsky, Helmut. Vektorrechnung in der Ebene und komplexe Analysis. Math.-Phys. Semesterber. 6 (1958), 46–56.

An exposition.

1999:

Vivier, Marcel. Étude des propriétés d'opérateurs différentiels par des méthodes d'algèbre extérieure directe. Ann. Sci. École Norm. Sup. (3) 74 (1957), 179–195.

L'Auteur établit, par des méthodes d'algèbre extérieure directe et une utilisation systématique du calcul matriciel, le théorème suivant: les composantes effectives du Laplacien d'une forme différentielle extérieure, décomposée en son polynôme de Lepage, sont les Laplaciens de ses composantes effectives. *G. Papy* (Bruxelles)

2000:

Petrov, P. I. Second order invariants of the quaternary differential quadratic form. Dokl. Akad. Nauk SSSR (N.S.) 113 (1957), 1214–1217. (Russian)

The author remarks that neither the work of E. Cartan [J. Math. Pures Appl. (9) 1 (1922), 141–203] nor that of N. A. Rozenson [Trudy Indust. Inst. Razdel Fiz.-Mat. 4 (1937), 59–84] contains an exhaustive treatment of the full system of the scalar differential invariants of the second order of a quadratic form and proceeds to the construction of this system by solving the system of 24 partial linear differential equations in 21 variables which these invariants have to satisfy. It is hardly avoidable, in this approach to the problem, that errors should occur, and indeed the copy sent to the reviewer was accompanied by a full-page letter from the author, indicating that only two formulas are correct from those given in the original paper.

It may be of use to add the following remarks: 1. The problem was treated completely in Weitzenböck, Invariantentheorie [Noordhoff, Groningen, 1923, Abschnitt

XIII, § 14]. Representing the Riemann tensor by the symbolical quadratic line complex $(a^2\pi^2)(\alpha^2\rho^2)$ and the fundamental tensor by $(g'x)^2$, giving rise to the tangential complex $(G^2\pi^2)(\Gamma^2\rho^2)$, the system is given by the 14 invariants

$$I_1=(a^2\alpha^2)=0,\ I_2=(a^2\beta^2)(b^2\alpha^2),$$
$$I_3=(a^2\beta^2)(b^2\gamma^2)(c^2\alpha^2),\ \cdots,\ I_6,$$

and the polar forms of these with respect to $G^2\cdot\Gamma^2$,

$$I_2'=R,\ I_3',\ I_4',\ I_4'',\ I_5',\ I_5'',\ I_6',\ I_6'',\ I_6'''.$$

This system gives therefore the required invariants directly in the tensor components. 2. In order to characterise the system obtained by the author, we remark that the tangential complex of a quadratic surface is of the type [(111)(111)], and taking six complexes, linear and in involution, $x_1, x_2, x_3, y_1, y_2, y_3$, we have $x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2=0$, whereas $x_1^2+x_2^2+x_3^2-y_1^2-y_2^2-y_3^2=0$ is the tangential complex. An arbitrary quadratic line complex can then be written

$$(A'x)^2+(B'y)^2+(C'x)(c'y)=0.$$

From this, it is clear that we have to find the ternary projective invariants of these last three forms, where x and y belong to different planes, adding the form $(F'x)^2=x_1^2+x_2^2+x_3^2$. We have the invariant $(C'D'E')(c'd'e')=\vartheta_2$, corresponding to the determinant of the matrix of the correlation, and the invariants of the system in which this correlation is replaced by $(c'F)(d'F)(C'x)(D'x)=(K'x)^2$ and $(C'F)(D'F)(c'y)(d'y)=(M'y)^2$. This system is well-known to be:

$$\sigma_0=(FA')^2,\ \sigma_1=(F'A)^2,\ \sigma_2=(AA')^2,\ \omega_3=(KA')^2,$$
$$\omega_2=(K'A)^2,\ \omega_1=(F'A'K')^2,$$
$$\kappa_0=(FB')^2,\ \kappa_1=(F'B)^2,\ \kappa_2=(BB')^2,\ \bar{\omega}_3=(MB')^2,$$
$$\bar{\omega}_2=(M'B)^2,\ \bar{\omega}_1=(F'B'M')^2,$$
$$\vartheta_1=(FK')^2=(FM')^2,\ \vartheta_3=(F'K)^2=(F'M)^2,$$
$$(KK')^2=(MM')^2\sim\vartheta_2^2.$$

For a Riemann tensor $(a^2\alpha^2)=\sigma_0+\kappa_0=0$, $\sigma_0-\kappa_0=R$, so σ_0 and κ_0 can be replaced by the curvature R.

The extensive and unsymmetrical formulas obtained by the author are those for which the matrices corresponding to $(A'x)^2$, $(B'y)^2$ and $(C'x)(c'y)$ are

$$\begin{Vmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{Vmatrix},\quad \begin{Vmatrix} b_1 & b_4 & b_5 \\ b_4 & b_2 & b_6 \\ b_5 & b_6 & b_3 \end{Vmatrix},\quad \begin{Vmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{Vmatrix}$$

with the restriction that his ω_1 and $\bar{\omega}_1$ are different by the terms $\sigma_0\vartheta_1$ and $\kappa_0\vartheta_1$ respectively, the subtraction of which would have simplified the results obtained by the author. *E. M. Bruins* (Amsterdam)

2001:

Švec, Alois. Les congruences de droites dans S_5. Publ. Fac. Sci. Univ. Masaryk **1957**, 87–100. (Russian summary)

E. Čech a introduit la notion de déformation ponctuelle des congruences de droites [Czechoslovak Math. J. **5(80)** (1955), 234–273; MR **17**, 658], et A. Švec a montré [ibid. **5(80)** (1955), 559–572; MR **18**, 230] que dans S_n, où $n\geq 5$, la déformation projective du second ordre des congruences équivaut à là déformation ponctuelle. Dans le présent mémoire, l'auteur définit la bidéformation projective dans S_5.

D'abord, à une congruence L de droites dans S_5, dont les deux surfaces focales possèdent chacune un seul réseau conjugué, on associe sa "dualisation", congruence engendrée dans l'espace dual par les espaces tangents à la congruence L le long de chacun de ses rayons. La forme ponctuelle de cette dualisation est appelée forme hyperplanaire de L et ceci conduit à une généralisation de l'invariant de Waelsch. Une correspondance développable entre deux congruences L et L' (c'est-à-dire une correspondance dans laquelle chaque développable de L est transformée en développable de L') est appelée bidéformation projective si elle réalise simultanément une déformation projective du second ordre entre L et L' et entre leurs dualisations. Il est démontré que les couples de congruences de droites en bidéformation existent et dépendent d'une fonction de deux variables.

M. Decuyper (Lille)

2002:

Nádeník, Zbyněk. La couche d'hypersurfaces et la correspondance nulle dans un espace projectif à n dimensions. Czechoslovak Math. J. **7(82)** (1957), 73–95. (Russian. French summary)

Im projektiven S_n wird durch $x_i=x_i(u_1, \cdots, u_{n-1}, v)$ eine von v abhängige Schar von Hyperebenen definiert. In einem bestimmten Gebiet $\Omega\subset S_n$ möge durch jeden Punkt (x_i) nur eine Hyperfläche der Schar gehen und dual dazu soll auch die Tangentialhyperebene in (x_i) an diese Scharhyperebene keine weitere der Schar berühren. Dann hat man in Ω eine Beziehung zwischen Punkten und hindurchgehenden Tangentialhyperebenen definiert, die man nach E. Čech sinnvoll Nullkorrelation N vom holonomen Typ nennt. Wie bereits die projektive Nullkorrelation lehrt, ist der in dem Namen holonom zum Ausdruck kommende Zusammenhang mit einer Hyperflächenschar, bei $n\geq 3$ eine Besonderheit. Die vorliegende Arbeit basiert weiterhin wesentlich auf den von E. Čech eingeführten Begriffsbildungen [Czechoslovak Math. J. **2(77)**, 91–107, 109–123, 125–148, 149–166, 167–188 (1952), 297–331 (1953); MR **16**, 71]. Es werden nach dem Muster von E. Cartan der Nullkorrelation N angepaßte bewegliche projektive Koordinatensysteme eingeführt und zugehörige Ableitungsgleichungen und Pfaffsche Formen. Vor allem ergibt sich daraus zunächst zu jedem Punkt A^0 von Ω ein ausgezeichnetes Quadrikenbüschel, das in geeigneten Koordinaten die Form $\sum_{i=1}^{n-1} x_i^2+2x_0x_n-2\lambda x_0^2=0$ hat. Alle diese Quadriken haben den Asymptotenkegel der durch A^0 gehenden Scharhyperfläche gemein. Dann werden die in A^0 im Sinne von Čech tangierenden Korrelationen berechnet. Diese hängen im allgemeinen von n Parametern ab. *W. Burau* (Hamburg)

2003:

Gambotto, Anna Maria. Sui sistemi semplicemente infiniti di spazi con incidenza di spazi generatori infinitamente vicini. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **91** (1956–57), 215–225.

This is an extension of the paper: A. Terracini, Scritti Mat. Off. a Luigi Berzolari, Pavia 1936, pp. 449–478. In a projective space S_{2h+1}, let a variable projective subspace S_h be generated by $(h+1)$ linearly independent points $a_0(t), a_1(t), \cdots, a_h(t)$, which are all analytic functions of the single parameter t. If t and $t_1\neq t$ define two different such subspaces S_h, the condition of incidence between them is that the determinant D of order $(2h+2)$ formed by the coordinates of the corresponding two sets of points be zero. Set $t_1=t+\alpha$. A simply infinite system of S_h is a system Δ if the determinant D is an infinitesimal of order $\rho>h+1$, with respect to α. The difference $\sigma=\rho-h-1$, is called the order of approximation of the incidence. If

$\sigma \neq \infty$, then $\sigma \leqq \binom{2h+2}{h+1} - h - 2$. When $\sigma \geqq 6$, the canonical form of the system of differential equations defining the set Δ is obtained. *J. De Cicco* (Chicago, Ill.)

2004:

Saveliev, S. I. Surfaces with plane generatrices along which the tangent plane is constant. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 663–665. (Russian)

Listed are some generalizations of developable surfaces to certain manifolds of projective N-space, obtained with the ω-symbolism of S. P. Finikov. One of the theorems states that in the case that the osculating planes at the points of an n-dimensional surface with p-dimensional generating planes ($n-p>1$), along which the tangent plane is constant, have dimension $>n+(n-p)$, then the surface is conical with a $(p-1)$-dimensional plane vertex. Fully classified is the case $p=n-2$ with osculating planes of dimension $n+2$ (the case $n+3$ is covered by the given theorem, the case $n+1$ gives surfaces in projective $(n+1)$-space). Among the references we find N. N. Janenko, Trudy Moskov. Mat. Obšč. **3** (1954), 89–180 [MR **16**, 70]. *D. J. Struik* (Cambridge, Mass.)

2005:

Laugwitz, Detlef. Eine Bemerkung über Flächenabbildungen. Jber. Deutsch. Math. Verein. **60** (1958), Abt. 1, 93–96.

The author studies a cartogram T between two surfaces S and $\bar{S}$ such that at corresponding points, the Christoffel symbols of the second kind Γ_{jk}^{i}, for $i, j, k=1, 2$, are equal. It is proved that either T is a homothetic map between S and $\bar{S}$, or S and $\bar{S}$ are both planes and T is an affine map. This result is not true in general for such a cartogram T between two Riemannian manifolds V_n and $\bar{V}_n$ for $n \geqq 3$. However, it is shown how this type of map T is related to a projective correspondence between V_n and $\bar{V}_n$ for $n \geqq 3$. *J. De Cicco* (Chicago, Ill.)

2006:

Kurita, Minoru. Geodesic correspondence of Riemann spaces. J. Math. Soc. Japan **8** (1956), 22–39.

Let S and $\bar{S}$ be two n-dimensional Riemann spaces which are in geodesic correspondence. The author defines the characteristic roots of this correspondence. The cases in which either (1) all the characteristic roots are equal or (2) all are equal except one are investigated in detail. Typical results in the first case are: If all the characteristic roots of a geodesic correspondence of $\bar{S}$ with S are equal then the roots are constant. If this constant is positive then, after a suitable choice of coordinates, the metric of S may be written as $ds^2=(dx^1)^2+\sin^2 x^1 \cdot g_{\alpha\beta}(x^2, \cdots, x^n)\, dx^\alpha dx^\beta$ except for a multiplicative constant. Among the spaces $\bar{S}$ which are in geodesic correspondence with S there is one with the metric $ds^2=\sec^4(x^1)(dx^1)^2+\tan^2 x^1 g_{\alpha\beta}(x^2, \cdots, x^n)dx^\alpha dx^\beta$. These two spaces are also in conformal correspondences, the point $(2x^1, x^2, \cdots, x^n)$ of $\bar{S}$ mapping on $(x^1, x^2, \cdots, x^n)$ of S. The geodesic and conformal correspondences of S, $\bar{S}$ by different transformations is a generalization of the central and stereographic projection of a sphere on a plane. When the characteristic roots are a negative constant or zero, similar results are obtained. Under the second case, there is a detailed discussion of the geodesic correspondence of the Riemann space S with metric $ds^2=(dx^1)^2+c(x^1)^2 \cdot g_{\alpha\beta}(x^2, \cdots, x^n)dx^\alpha dx^\beta$ with other spaces.

A. Fialkow (Brooklyn, N.Y.)

2007:

Kanitani, Jôyô. Sur la forme de Darboux généralisée. II, III. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **29** (1955), 229–248; **30** (1956/57), 43–56.

In these two papers the author continues to study the same problem as in part I [same Mem. **28** (1954), 225–239; MR **16**, 516]. Suppose that the matrix

$$\mathfrak{M}(Q)=\begin{pmatrix} Q^i R_{i112} & \cdots & Q^i R_{i1n-1,n} \\ \cdot & \cdots & \cdot \\ Q^i R_{inn2} & \cdots & Q^i R_{inn-1,n} \end{pmatrix}$$

is of rank $n-2$, where $i=1, \cdots, n$, and that the equation

$$D(\rho)\equiv \begin{vmatrix} \mu_3{}^3-\rho & \cdots & \mu_n{}^3 \\ \cdot & \cdots & \cdot \\ \mu_3{}^n & \cdots & \mu_n{}^n-\rho \end{vmatrix} = 0$$

admits e distinct roots $\rho_1, \cdots, \rho_e$ of orders $\lambda_1, \cdots, \lambda_e$, respectively, and the determinants $D(\rho_1), \cdots, D(\rho_e)$ are of ranks $n-2-h_1, \cdots, n-2-h_e$, respectively, where $\mu_h{}^j$ are given by the following equations, obtained in part I:

$$R_{2hlm}=\sum_{j=3}^{n} \mu_h{}^j R_{1jlm} \quad (h=3, \cdots, n;\ l, m=1, \cdots, n).$$

Part II deals with the case $e \geqq 3$, $h_s=\lambda_s$ $(s=1, \cdots, e)$, while part III deals with the case $e=2$, $h_s=\lambda_s$ $(s=1, 2)$.

C. C. Hsiung (Bethlehem, Pa.)

2008:

Prvanović, Mileva. Les espaces parageódésiques et les courbes parageódésiques appartenant au sous-espace d'espace riemannien. Srpska Akad. Nauka. Zb. Rad. **50**, Mat. Inst. **5** (1956), 117–178. (Serbo-Croatian. French summary)

Let $\mathscr{V}_m$ be a subspace of a Riemannian space $\mathscr{V}_l$ and let in $\mathscr{V}_l$ be given a set of $l-m$ congruences of curves such that through every point in $\mathscr{V}_m$ passes one curve of each congruence. Let $\lambda_{r|}{}^\alpha$ be contravariant components of a unit vector tangent to the curve of congruence $\lambda_{r|}$, where index r fixes the congruence in the system of $l-m$ congruences passing through a given point in $\mathscr{V}_m$. The author defines as parageodesic subspace of $\mathscr{V}_m$ in Riemannian space $\mathscr{V}_l$, with respect to the congruence $\lambda_{r|}$, the subspace $\mathscr{V}_n$ having the property that in every point of $\mathscr{V}_n$ its vector of normal curvature, with respect to $\mathscr{V}_l$ and in any direction e^i in $\mathscr{V}_n$, is collinear with the tangent vector $\lambda_{r|}{}^\alpha$ of the congruence $\lambda_{r|}$ in the corresponding point. Differential equations of such spaces are derived and it is demonstrated that the totally geodesic subspaces are a particular case of parageodesic spaces.

In case $n=1$, the parageodesic space $\mathscr{V}_n$ is a parageodesic curve in $\mathscr{V}_m$ with respect to $\lambda_{r|}$. Among other properties, it is demonstrated that: 1) A parageodesic curve in $\mathscr{V}_m$ is a curve whose first curvature vector with respect to $\mathscr{V}_l$, in every point, is collinear with the tangent vector $\lambda_{r|}{}^\alpha$ in the corresponding point in $\mathscr{V}_m$; 2) parageodesic curves are a generalisation of geodesics in $\mathscr{V}_m$; 3) the unit tangent vectors of a parageodesic curve constitute the field of autoparallel vectors along the curve with respect to a specially defined linear connection. For curves which are not parageodesics, in every point there is defined a parageodesic curvature vector with respect to the congruence $\lambda_{r|}$. In connection with that, Euler's formula and Meusnier's theorem are generalized.

T. P. Anđelić (Belgrade)

2009:

Stojanovitch, Rastko. Motion of a rigid body in two-dimensional Riemannian space. Glas Srpske Akad. Nauka **221**, Od. Prirod.-Mat. Nauka (N.S.) **9** (1956), 63–73. (Serbo-Croatian. English summary)

The author defines a rigid body in the two-dimensional Riemann space V_2. Such a body has at most three degrees of freedom. The case in which it has exactly three degrees of freedom occurs when V_2 is flat or of constant gaussian curvature. Then both translation and rotation can be defined in nearly the same way as in the euclidian space. None of the spaces V_2 admits two degrees of freedom only, and one degree of freedom is possible when metrics of V_2 are of a special form.

From the general system of equations of motion in a field of force, the author derives differential equations of motion for a rigid body in V_2. *T. Leser* (Aberdeen, Md.)

2010:

Stojanović, Rastko. Motion of a rigid body in Riemannian spaces of constant curvature. Srpska Akad. Nauka. Zb. Rad. **50**, Mat. Inst. **5** (1956), 219–238. (Serbo-Croatian. English summary)

The author defines a rigid body in the n-dimensional Riemann space K_n and connects the problem of this body with the theory of groups of motions. The Riemann space of constant curvature admits a complete group of motions G_r of order $r=\frac{1}{2}n(n+1)$; hence a free rigid body in K_n has $\frac{1}{2}n(n+1)$ degrees of freedom. The group G_r has $\frac{1}{2}n(n-1)$ parameters of orientation of the body, and n parameters of position. In the case of rotation about a fixed point in K_n, the body has $\frac{1}{2}n(n-1)$ degrees of freedom, and the rotation consists of $\frac{1}{2}n(n-1)$ mutually independent motions about $\frac{1}{2}n(n-1)$ geodetic subspaces of K_n, each subspace being $(n-2)$-dimensional. The necessary, but not sufficient, condition for pure translation is that the orientation of the body be fixed; the additional condition is that K_n must be flat or the fundamental form indefinite, because in general the space K_n does not admit parallel vector fields.

Using Lie's first fundamental theorem on continuous groups, the author derives expressions for infinitesimal displacements, in which the parameters of the group G_r appear as independent coordinates of the body and also of the differential equations of motion.

T. Leser (Aberdeen, Md.)

2011:

Vrănceanu, G. Tenseurs harmoniques et groupes de mouvements d'un espace de Riemann. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 257–277. (Romanian. Russian and French summaries)

There exists a relationship between the number r of the parameters of the group of motions of a compact Riemannian V_n and its Betti number B_1. It is possible to find maxima for r if an invariant closed vector or an invariant closed bivalent tensor field is given, especially if B_1 or B_2 is not zero. Certain formulas are derived which connect the closed invariant tensor fields to the G_r. If the group is a simply transitive G_n the search for harmonic tensors becomes an algebraic problem. Here a result of W. Hodge [Theory and applications of harmonic integrals, Cambridge Univ. Press, 1952; MR **14**, 500; esp. p. 255] is generalized and conditions are found for the existence of a certain number of harmonic vectors and bivalent tensors. *D. J. Struik* (Cambridge, Mass.)

2012:

Mishra, R. S. Non holonomic subspaces. Riv. Mat. Univ. Parma **7** (1956), 349–357.

T. Takasu [Yokohama Math. J. **1** (1953), 1–28, 29–38, 39–74, 75–77, 79–82, 83–87; MR **15**, 350] has treated the space based on the metric: $ds^2=G_{mn}(x^\lambda)w^m w^n$ $(w^l=w_\mu{}^l(x^\lambda)dx^\mu)$, discovering that the paths (II-geodesics) $d^2x^\lambda/ds^2+\Lambda_{\mu\nu}{}^\lambda(dx^\mu/ds)(dx^\nu/ds)=0$ (with $\Lambda_{\mu\nu}{}^\lambda=\Omega_l{}^\lambda\partial w_\mu{}^l/\partial x^\nu=-w_\mu{}^l\partial\Omega_l{}^\lambda/\partial x^\nu$, $\Omega_n{}^\lambda w_\lambda{}^m=\delta_n{}^m$, $\Omega_l{}^\lambda w_\mu{}^l=\delta_\mu{}^\lambda$; $\lambda, \mu, \nu, \cdots; l, m, n, \cdots=1, 2, \cdots, N$) of teleparallelism behave as for meet and join like straight lines. He adopted the coordinates $\xi^l=\int(w^l/ds_l)ds_l=a^l s_l+c^l$ $(G_{mn}a^m a^n=1)$, referred to II-geodesic coordinate axes, so that $\Omega_l{}^\nu\partial/\partial x^\nu=\partial/\partial w^l=\partial/\partial\xi^l$. The present author introduces "Non-Holonomic" (NH) curvature tensor, NH-Ricci tensor, the normal to NH-hypersurface, and then generalizes the Mainardi-Codazzi equations in the NH-hypersurface. Thereby the Christoffel symbols are constructed from G_{mn}. [Cf. G. Vrănceanu, C. R. Acad. Sci. Paris **183** (1926), 852–854, 1083–1085; Z. Horak, Publ. Fac. Sci. Univ. Masaryk, no. **86** (1927); J. A. Schouten, Math. Z. **30** (1929), 149–172.]

T. Takasu (Yokohama)

2013:

Yastrebov, Yu. M. Pairs of connections with geodesic parallelograms. Černivec. Derž. Univ. Nauk. Zap. Ser. Fiz.-Mat. **19** (1956), no. 4, 109–123. (Ukrainian. Russian summary)

2014:

Auslander, Louis. On the group of affinities of locally affine spaces. Proc. Amer. Math. Soc. **9** (1958), 471–473.

Let M be a compact manifold with a given complete flat affine connection (i.e. curvature and torsion zero), and G the group of all affinities of M, i.e., the group of all homeomorphisms of M onto itself which preserve the given affine structure on M. It was proved by Nomizu [Proc. Amer. Math. Soc. **4** (1953), 816–823; MR **15**, 468] that G is a Lie group. The author proves that the identity component of G is a nilpotent Lie group. In order to prove it, he uses the theorem of Eilenberg-MacLane on groups operating on acyclic spaces [Ann. of Math. (2) **46** (1945), 480–509; MR **7**, 137] and the well-known fact that a linear Lie algebra, all of whose elements are nilpotent, is nilpotent. *A. Kawaguchi* (Sapporo)

2015:

Horváth, J. I. Zur Geometrisierung des elektromagnetischen Feldes. Nuovo Cimento (10) **7** (1958), 636–648.

Eine Parallelübertragung D heißt metrisch, wenn $Dg^{ik}=0$. Dabei bedeutet g^{ik} den kontravarianten metrischen Fundamentaltensor eines Riemannschen Raumes. Für gewöhnlich wird D für die Parallelübertragung kontravarianter Vektoren mit

$$(*) \qquad D\xi^i=d\xi^i+\Gamma^i{}_{rs}\xi^r dx^s=0,$$

identifiziert, wobei $\Gamma_{ijk}=\frac{1}{2}\{\partial_k g_{ij}+\partial_i g_{kj}-\partial_j g_{ik}\}$. D kann durch D^* gemäß

$$D^*\xi^i=d\xi^i+L^i_{rs}\xi^r dx^s,\quad L_{rjs}=\Gamma_{rjs}+\Lambda_{rjs}$$

ersetzt werden und die Ableitungen Übertragung bleibt metrisch, wenn $\Lambda_{rjs}=-\Lambda_{jrs}$. Mit L^i_{rs} entsteht jetzt der Krümmungstensor eines quasi-Riemannschen Raumes

$$(**) \qquad T_i{}^j{}_{km}=R_i{}^j{}_{km}+P_i{}^j{}_{km},$$

wobei $R_i{}^j{}_{km}$ den gewöhnlichen gemischten Riemann-Christoffelschen Krümmungstensor bezeichnet. Für (**) gewinnt Verfasser nunmehr Analogien zu den Identitäten von Ricci und Bianchi. Von Bedeutung wird auch der

Unterschied der Autoparallelen

$$(***) \qquad \frac{d^2x^i}{ds^2}+L^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds}=0,$$

$$(****) \qquad \frac{d^2x^i}{ds^2}+\Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds}=0.$$

Der Übergang zur Physik erfolgt durch den Ansatz $\Lambda_{ijk}=F_{ij}s_k$, $F_{ij}=-F_{ji}$ elektromagnetischer Feldtensor, $s^k=e(dx^k/ds)$ Viererstrom. Aus (***) ergibt sich jetzt die Bewegungsgleichung geladener Massenpunkte und die Lorentzsche Kraftdichte bestimmt unmittelbar die Abweichung von (***) gegenüber (****). Damit ist eine sehr einfache Geometrisierung des elektromagnetischen Feldes gewonnen. Die Feldgleichungen der neuen Theorie lassen sich aus einem Variationsprinzip ableiten. Durch die elektromagnetischen Felder bleibt die gewöhnliche (Riemannsche) Krümmung unbeeinflußt. Auch die Einsteinschen Gravitationseffekte bleiben in der neuen Theorie ungeändert. *M. Pinl* (Köln)

2016:

Vrănceanu, G. Sur les invariants intrinsèques d'un espace non holonome. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956) no. 11, 9–23. (Romanian. Russian and French summaries)

The intrinsic invariants of a non-holonomic V_n^m are the invariants belonging to the Pfaff system $d\bar{s}^h=c_k^h ds^k+c_\alpha^h ds^\alpha$, $d\bar{s}^\alpha=c_\beta^\alpha ds^\beta$ ($h, k=1, \cdots, m$; $\alpha, \beta=m+1, \cdots, n$), where c_k^h is an orthogonal matrix. When the system $ds^\alpha=0$ has derived systems the given Pfaff system can be arranged in such a way that its matrix takes the triangular form M_{ab}, $a, b=1, \cdots, l$; $M_{ab}=0$, $a>b$. Here M_{11} is the matrix c_k^h and the other diagonal terms $M_{22}, \cdots, M_{l-1,l-1}$ are also orthogonal matrices. The matrix M_{ll} is the square matrix which transforms the integrable combinations of the given Pfaff system. The case where $ds^\alpha=0$ has no derived systems has already been taken up by M. Haimovici [J. Math. Pures Appl. **25** (1946), 291–305; MR **9**, 467]. *D. J. Struik* (Cambridge, Mass.)

2017:

Simionescu, C. La détermination des invariants des espaces nonholonomes V_5^3 de classe cinq et d'espèce trois. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 12, 33–39. (Romanian. French and Russian summaries)

Referring to a theory of G. Vrănceanu [see review above] the author shows that the V_5^3 ($ds^\alpha=\lambda_i^\alpha dx^i$, $\alpha=4, 5$; $i=1, \cdots, 5$) of class 5 and rank ("espèce") 3 (here the bilinear covariant $\Delta s^1=\alpha[ds^2ds^3]$ (mod. ds^1, ds^4, ds^5) has $\alpha=0$) admit a complete system of invariants. *D. J. Struik* (Cambridge, Mass.)

2018:

Gheorghiu, Octavian Em. Sur les objets géométriques associés à un système linéaire d'équations aux dérivées partielles du premier ordre. C. R. Acad. Sci. Paris **247** (1958), 26–28.

On énumère des types d'objets géométriques associés à un certain système en considération du groupe conforme de S_3. *Résumé de l'auteur*

2019:

Lemlein, V. G. Spaces of symmetric almost symplectic connectivity. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 655–658. (Russian)

Spaces of symmetric symplectic connectivity have a $\Gamma_{jk}{}^l=\Gamma_{kj}{}^l$ and a nondegenerated skew symmetric fundamental tensor a_{ij} such that $T_{ijk}=\nabla_{[k}a_{ij]}=0$. In the present case of almost symplectic connectivity $T_{ijk}=\nabla_k a_{ij}$. Then $\Gamma_{jk}{}^l=\frac{1}{2}a^{lt}(\partial_k a_{tj}-\partial_t a_{kj}+\gamma_{tjk})$, $a^{lt}a_{ik}=\delta_k{}^l$, where the γ form a geometrical object for which

$$\gamma_{i'j'k'}=\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}\gamma_{ijk}+a_{pq}\frac{\partial x^p}{\partial x^{(i'}}\frac{\partial^2 x^q}{\partial x^{j'}\partial x^{k')}}.$$

The number of dimensions of such spaces must be even. When $b_i=\frac{1}{2}a^{pq}T_{pqi}$, then $b_i=-\partial_i \ln\sqrt{a}$, $a=|a_{ij}|$. Let $R_{ijk}{}^{\cdot\cdot\alpha}$ be the curvature tensor built with the $\Gamma_{ij}{}^l$, then "flat" spaces are defined by $B_{ij,kl}=a_{l\alpha}R_{ijk}{}^{\cdot\cdot\alpha}=0$. In such spaces a cartesian coordinate system can be found for which $a_{ij}=\alpha_{ijk}x^k+\beta_{ij}$, where the α and β are constants, $\alpha_{ijk}=T_{ijk}$. *D. J. Struik* (Cambridge, Mass.)

2020:

Milnor, John. On the existence of a connection with curvature zero. Comment. Math. Helv. **32** (1958), 215–223.

Let M^2 be a closed oriented C^∞ surface of genus g and let $GL^+(2)$ denote the group of 2×2 real matrices with positive determinant. To each principal fiber bundle with structure group $GL^+(2)$, called a $GL^+(2)$ bundle, over M^2, there corresponds the Euler class $X\in H^2(M^2, Z)$. Let $X[M^2]$ denote the Kronecker index of X with the fundamental cycle of M^2. The author then proves the following two theorems. Theorem 1: If $|X[M^2]|\geqq g>0$, then the $GL^+(2)$-bundle over M^2 with Euler class X does not possess a connection with curvature zero. Theorem 2: If $|X[M^2]|<g$, then the $GL^+(2)$ bundles over M^2 with Euler class X does have a connection with curvature zero. The first theorem has the following corollary. A surface with genus $g\geqq 2$ does not have any affine connection with curvature zero (symmetric or not). *L. Auslander* (Bloomington, Ind.)

2021:

Fabian, William. Tensor integrals. Proc. Edinburgh Math. Soc. (2) **10** (1957), 145–151.

The tensor integral of a tensor along a curve in a V_n is defined as the inverse of the intrinsic derivative with respect to an affine connection in the V_n. Certain simple theorems concerning tensor integration and differentiation are then developed, including integration by parts and an extension of Taylor's theorem applied to tensors. *A. J. McConnell* (Dublin)

2022:

Satake, Ichiro. The Gauss-Bonnet theorem for V-manifolds. J. Math. Soc. Japan **9** (1957), 464–492.

This paper generalizes the Gauss-Bonnet theorem as proved by Allendoerfer and Weil to V-manifolds. For the definition of a V-manifold see Satake, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363 [MR **18**, 144]. An example of a V-manifold is Siegel's modular variety $\mathfrak{B}_n=\mathbf{M}_n/\mathfrak{H}_n$ where $\mathfrak{H}_n$ is the space of all symmetric complex matrices $Z=X+iY$ of degree n with $Y>0$; and $\mathbf{M}_n$ is the group of all sympletic transformations of degree $2n$ with integral coefficients operating on $\mathfrak{H}_n$. A characteristic feature of a V-manifold is the existence of a set of dim. $\leqq n-2$ of singular points; i.e. points with non-trivial isotropy groups. If this set is empty, the V-manifold is an ordinary manifold.

The author extends to V-manifolds the usual differential structures of a manifold such as: fibre bundles, vector fields, differential forms, Riemannian metric, connection, and curvature. The index $I(p)$ of a unit

vector field at a singular point, p, of the field is defined to be the quotient of its usual index by the order of the isotropy subgroup at the point. Let Ω be the usual integrand in the Gauss-Bonnet Theorem, then it is proved that for a compact, orientable, Riemannian V-manifold M_n: $\int_M \Omega = (-1)^n \sum_i I(p_i)$. The expression $\sum_i I(p_i)$ is called $\chi_V(M)$, the Euler characteristic of M as a V-manifold. It is not necessarily an integer, and generally differs from $\chi(M)$. If M has an appropriate triangulation, $\chi_V(M)$ can be given a combinatorial interpretation.

These results are applied to Siegel's $\mathfrak{B}_n$. Siegel restricted himself to the case where $\mathbf{M}_n \backslash \mathfrak{H}_n$ is compact and where $\mathbf{M}_n$ is a discontinuous group without fixed point. In this paper both of these restrictions are removed. Finally an estimate is given of the least common multiple of the orders of isotropy subgroups of $\mathbf{M}_n \backslash (\pm E_{2n})$.

C. B. Allendoerfer (Seattle, Wash)

2023:

Lelong-Ferrand, Jacqueline. Sur les transformations infinitésimales d'une variété différentiable, considérées comme des opérateurs hilbertiens. C. R. Acad. Sci. Paris **245** (1957), 1585–1588.

Let V^n be a manifold with a regular boundary (i.e. defined locally by $x_n=0$ in an admissible system of coordinates), let ξ be a vector field on V^n, and let $d\tau$ be a measure on V^n, all satisfying mild regularity conditions. Let $\mathscr{H}$ be the Hilbert space of all square-integrable functions, $\mathscr{H}_p^q$ the Hilbert space of all square-integrable p-covariant and q-contravariant tensors. Then ξ determines an operator X_ξ on $\mathscr{H}$ (or $\mathscr{H}_p^q$) on a domain consisting of regular functions with compact support. The first result stated is that if the divergence (defined with respect to $d\tau$) of ξ is bounded, then ξ determines a global one-parameter group of homeomorphisms of V^n if and only if the closure (strong extension) of X_ξ on $\mathscr{H}$ coincides with its weak extension (Hilbert space adjoint of its formal adjoint). If V^n is Riemannian, then ξ determines a group of isometries if and only if the closure of iX_ξ on some $\mathscr{H}_p^q$ with $n \neq 2(p-q)$ is self-adjoint. If $n=2(p-q)$, "isometries" is replaced by "conformal transformations."

The author states a sufficient condition for the weak and strong extensions to coincide. In case V^n is Riemannian, she states necessary and sufficient conditions for ξ to determine a compact one-parameter group, and, in case $d\tau$ is preserved, for $\mathscr{H}$ to be decomposed into the null space and range of the weak extension of X_ξ.

E. Nelson (Princeton, N.J.)

2024:

Ammann, André. Sur l'immersion d'une variété différentiable dans l'espace euclidien. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 828–833.

"Le théorème d'immersion de H. Whitney est établi dans cet article par une méthode voisine de celle indiquée par M. G. de Rham dans son livre sur les Variétés différentiables, et probablement aussi d'une démonstration (non publiée) de N. Bourbaki. Nous utilisons d'ailleurs un autre théorème du même auteur qui est en fait la clé de notre démonstration." (Résumé de l'auteur.)

L. Auslander (Bloomington, Ind.)

2025:

Ohmann, D. Eine Verschärfung der isoperimetrischen Ungleichung in der hyperbolischen Ebene. Arch. Math. **8** (1957), 355–359.

On the hyperbolic plane, let W be the measure (in the sense of integral geometry) of the set of lines which intersect a given set of points A of area F and Minkowski length U. The author proves (following a rather complicated way), that a) $W \leq U$, and b) $W^2 \geq F(4\pi+F)$.

L. A. Santaló (Buenos Aires)

2026:

Ostrowski, Alexander. Über die Evoluten von endlichen Ovalen. J. Reine Angew. Math. **198** (1957), 14–27.

A finite oval S_{2m} in the euclidean plane has a continuous never vanishing curvature and exactly $2m$ vertices. A straightforward application of a preceding paper [Arch. Math. **6** (1955), 170–179; MR **17**, 295] yields a description of the evolute Σ_{2m} of S_{2m}. A more detailed discussion of Σ_{2m} enables the author to prove the Four-Vertex-Theorem and to describe the possible shapes of the Σ_4's. If Σ_{2m} is simple, $m=2$. As another application, some results due to Schilling on curves of constant width are obtained and extended under weaker assumptions [Z. Math. Phys. **63** (1915), 67–136].

P. Scherk (Saskatoon, Sask.)

2027:

Papy, Georges. Variétés différentielles (point de vue contingent). Bull. Soc. Math. France **85** (1957), 1–14.

A discussion of manifolds, their tangent spaces and differentials and associated algebra, under highly general conditions.

W. Ambrose (Cambridge, Mass.)

2028:

Bielecki, Adam. Remarque à propos de la note "Certaines propriétés topologiques des solutions des équations au paratingent". Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **10** (1956), 95–97 (1958). (Polish and Russian summaries)

The author shows by a simple example that an hypothesis contained in Theorem 2 of the previous paper [same Ann. **9** (1955), 63–79; MR **19**, 549] is essential for the validity of this theorem.

L. Cesari (Baltimore, Md.)

PROBABILITY

See also 2065, 2068, 2069, 2109, 2110, 2111, 2113.

2029:

Wigner, Eugene P. On the distribution of the roots of certain symmetric matrices. Ann. of Math. (2) **67** (1958), 325–327.

Let v be a real symmetric matrix of large order N having random elements v_{ij}. Suppose that for $i \leq j$ the v_{ij} are independently distributed with even densities, equal second moments m^2 and nth moments bounded by constants B_n independent of i, j, N. For real $\alpha<\beta$ let $S = S_{\alpha,\beta}(v, N)$ be the number of latent roots of v that lie in the interval $(\alpha N^{\frac{1}{2}}, \beta N^{\frac{1}{2}})$. The author states that $\lim_{N\to\infty} N^{-1}E(S) = (2\pi m^2)^{-1}\int_\alpha^\beta (4m^2-x^2)^{\frac{1}{2}}dx$. The paper ends with the following statement. "The heuristic proof given for the special case given before applies equally under the more general conditions here specified." [Cf. Ann. of Math. (2) **62** (1955) 548–564; MR **17**, 1097.]

J. G. Wendel (Ann Arbor, Mich.)

2030:

Postnikov, A. G. The strong law of large numbers for samples from uniformly distributed random variables. Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 433–438. (Russian)

It is shown that, if $\alpha_1, \alpha_2, \cdots$ is almost any sample sequence from a sequence of mutually independent random variables, each uniformly distributed on the unit

interval [0, 1], then, for every strictly positive integer s, the sequence $\{\alpha_k, \cdots, \alpha_{k+s-1}, k \geqq 1\}$ is uniformly distributed on the unit s-dimensional interval. {Note by rev.: This is an immediate consequence of the strong law of large numbers for a stationary sequence of random variables.} *J. L. Doob* (Urbana, Ill.)

2031:

Cogburn, Robert. Lois limites des termes variationnels des sommes normées. C. R. Acad. Sci. Paris **246** (1958), 3408–3410.

The ranking limit problem was satisfactorily solved by Gnedenko [Ann. of Math. (2) **44** (1943), 423–453; MR **5**, 41] and Smirnov [Trudy Mat. Inst. Steklov **25**, 60 pp. 1949; MR **11**, 605] in the case of normed independent and identically distributed random variables, and by the reviewer (Proc. Third Berkeley Symposium, vol. 2, Univ. California, Berkeley, 1956, pp. 174–194; MR **18**, 942] in the case of double sequences of random variables. An important gap remained: the case of normed but not necessarily identically distributed random variables. The author states results which answer the various questions which appear in this case, thus filling the gap. {He mentions the following misprints: p. 3408, line 14, X in lieu of O; p. 3409, line 14, omit x; line 7 from bottom, "non-dégénérées" in lieu of "dégénérées".}

M. Loève (Berkeley, Calif.)

2032:

Rozanov, Yu. A. On the linear interpolability of stationary processes in discrete time. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 923–926. (Russian)

If f_λ is the spectral density of a stationary process in discrete time a necessary and sufficient condition for its linear interpolability is, according to A. N. Kolmogorov [Byull. Moskov. Gos. Univ. Mat. **2** (1941), no. 6; MR **5**, 101], that $\int_{-\pi}^{\pi} d\lambda / f_\lambda = \infty$. Using Hilbert space methods the author generalizes this condition to the case in which the stationary process X_t in discrete time takes on vector values. Let A be a separable Hilbert space and suppose that the values of the process X_t lie in A. Let $x_t(a) = (X_t, a)$ where $a \in A$. Suppose that the spectral functional $F_\lambda(a, b)$ of x_t has a density $f_\lambda(a, b)$. If $f_\lambda(a, b)$ is a bounded bilinear functional on A for almost all λ, then there is defined a bounded positive operator f_λ by the equality $f_\lambda(a, b) = (f_\lambda a, b)$. Let $\theta(\lambda) = \{a | f_\lambda a = 0\}$. Then $A(\lambda) = A \ominus \theta(\lambda)$. f_λ^{-1} exists on $A(\lambda)$. Let T be a finite set of integers. Let $B(T)$ be the Hilbert space of functions b_λ defined almost everywhere on $[-\pi, \pi]$ and such that (1) $b_\lambda = \sum_{t \in T} e^{i\lambda t} b_t$, $b_t \in A$, (2) $b_\lambda \in f_\lambda A$, (3) $\int_{-\pi}^{\pi} (f_\lambda^{-1} b_\lambda, b_\lambda)\, d\lambda < \infty$. X_t can then be interpolated in a mean-square sense on T, i.e. represented as a linear combination of values of X_t, $t \notin T$, if and only if $\dim B(T) = 0$. The generalization of Kolmogorov's condition is then found in (3) above.

H. P. Kramer (Murray Hill, N.J.)

2033:

Parzen, Emanuel. Conditions that a stochastic process be ergodic. Ann. Math. Statist. **29** (1958), 299–301.

The main result is as follows. Let $x(t)$ be a strictly stationary stochastic process and introduce for given time points $t_1, t_2, \cdots, t_k$ the characteristic function $\varphi(u_1, u_2, \cdots, u_k)$ of the stochastic variables $x(t_1), x(t_2), \cdots, x(t_k)$ and another characteristic function $\varphi(u_1, u_2, \cdots, u_k; \tau)$ of the increments $x(t_1) - x(t_1 + \tau), x(t_2) - x(t_2 + \tau), \cdots, x(t_k) - x(t_k + \tau)$. Then it is necessary and sufficient for $x(t)$ to be ergodic that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\tau=0}^{n} \varphi(u_1, u_2, \cdots, u_k; \tau) = |\varphi(u_1, u_2, \cdots, u_k)|^2$$

for every choice of $k, t_1, t_2, \cdots, t_k, u_1, u_2, \cdots, u_k$. This is a condition for asymptotic independence in a certain sense. *U. Grenander* (Stockholm)

2034:

Ciucu, George. Propriétés ergodiques de certaines chaînes à liaisons complètes. Acad. R. P. Romîne. Stud. Cerc. Mat. **8** (1957), 413–446. (Romanian. Russian and French summaries)

Dans ce travail, divisé en deux chapitres, l'auteur étudie certaines catégories de chaînes à liaisons complètes [voir O. Onicescu et G. Mihoc, Bull. Sci. Math. **59** (1935), 174–192, et W. Doeblin et R. Fortet, Bull. Soc. Math. France **65** (1937), 132–148].

Dans le premier chapitre on considère des chaînes ayant un ensemble dénombrable d'états possibles. Soient: (1) S un ensemble dénombrable; (2) $W = \prod_{j \in -N} S^j$ où $S^j = S$ pour tout $j \in -N = \{0, -1, \cdots\}$; (3) pour toute partie $I \subset N^* = \{1, 2, \cdots\}$, $S^I = \prod_{j \in I} S^j$ où $S^j = S$ pour $j \in I$. Pour toute suite $(x_1, \cdots, x_t) \in S^{(1, \cdots, t)}$ désignons par $u_{x_1, \cdots, x_t}$ l'application de W dans W définie par l'égalité: $u_{x_1, \cdots, x_t}(c) = (\cdots, x_{-n}, \cdots, x_{-1}, x_0, x_1, \cdots, x_t)$, si $c = (x_{-n})_{n \in N}$, $N = \{0, 1, \cdots\}$. Soient $P_x^n(c)$ ($c \in W$, $x \in S$, $n \in N$) les probabilités de passage d'une chaîne à liaisons complètes; pour tout $n > 1$ on a alors $P_x^n(c) = \sum_{y \in S} P_y^1(c) P_x^{n-1}(u_y(c))$. Pour $c \in W$ et $A \subset S^{(1, \cdots, t)}$ posons

$$P_A^1(c) = \sum_{(a_1, \cdots, a_t) \in A} P_{a_1}^1(c) P_{a_2}^1(u_{a_1}(c)) \cdots P_{a_t}^1(u_{a_1, \cdots, a_{t-1}}(c));$$

posons encore, pour $n > 1$, $P_A^n(c) = \sum_{y \in S} P_y^1(c) P_A^{n-1}(u_y(c))$. La chaîne est: 1° du type (B) si quels que soient $c, c' \in W$, $y \in S$ et $(x_1, \cdots, x_t) \in S^{(1, \cdots, t)}$ on a $P_y^1(u_{x_1, \cdots, x_t}(c)) = P_y^1(u_{x_1, \cdots, x_t}(c'))(1 + \theta\varepsilon_t)$, où $|\theta| = |\theta(c, c', y, s)| \leqq 1$, $\varepsilon_s > 0$ et $\sum_{s \in N^*} \varepsilon_s < \infty$; 2° du type (G) s'il existe une fonction P_A^∞ définie pour toute $A \subset S^{(1, \cdots, t)}$, $t \in N^*$ et deux constantes L et $\lambda > 0$ qui vérifient l'inégalité $|P_A^n(c) - P^\infty(A)| \leqq L e^{-\lambda \sqrt{n}}$ pour $n \in N^*$, $c \in W$ et $A \subset S^{(1, \cdots, t)}$, $t \in N^*$. Désignons par T la tribu engendrée par les parties $\prod_{j \in N^*} A_j$ où $A_j \subset S$ et pour toute $A = \mathrm{pr}^{-1}_{(1, \cdots, t)}(A_1) \in T$ posons $Q_c'(A) = P_{A_1}^1(c)$; désignons par Q_c l'unique extension complètement additive de Q_c' à T. La valeur moyenne d'une variable aléatoire intégrable h définie sur l'espace probabilisé $\{S^{N^*}, T, Q_c\}$ sera notée $M_c(h)$. Pour toute fonction bornée f définie sur S et $n \in N^*$ soit $f_n = f \circ \mathrm{pr}_n$ et $g_n = \sum_{j=1}^{n} f_j$. Posons encore $M_1^\infty = \sum_{x \in S} f(x) P_{(x)}^\infty$, $M_2^\infty = \sum_{x \in S} f(x)^2 P_{(x)}^\infty$ et pour $t > 2$

$$M(f_1 f_t) = \sum_{(a_1, \cdots, a_t) \in S^{(1, \cdots, t)}} f(a_1) f(a_t) P^\infty_{((a_1, \cdots, a_t))}.$$

Ces notations étant introduites on peut formuler les théorèmes suivants qui constituent les principaux résultats du premier chapitre: (I) Considérons une chaîne du type (B) dont les probabilités de passage vérifient la condition: il existe $\lambda > 0$ telle que $P_x^1(c) \geqq \lambda P_x^1(c')$ pour $c, c^1 \in W$ et $x \in S$. Alors il existe une suite $(\alpha_n)_{n \in N^*}$ convergeant vers 0 et une fonction P_A^∞, définie pour toute partie $A \subset S^{(1, \cdots, t)}$, $t \in N^*$, vérifiant l'inégalité

$$|P_A^n(c) - P_A^\infty| \leqq \alpha_n$$

quels que soient $n \in N^*$, $c \in W$ et $A \subset S^{(1, \cdots, t)}$. (II) Soit $\sigma^2 = M_2^\infty + \sum_{u \in N^*} M_1(f_1 f_{t+1})$ (on suppose $M_1^\infty = 0$). Alors il existe une constante H telle que

$$|M_c(g_n^2)/n - \sigma^2| \leqq H((1/n) + \inf_{q \in N^*} ((q+1)/n + e^{-\lambda \sqrt{(q+1)}}))$$

quels que soient $n \in N^*$ et $c \in W$. (III) Si $\sigma^2 \neq 0$ on a pour tout $c \in W$,

$$\lim_{n \to \infty} Q_c(g_n / \sqrt{n} < \sigma a) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{a} \exp(-u^2/2)\, du.$$

Dans le second chapitre l'auteur étudie certaines chaînes à liaisons complètes dont l'ensemble d'états possibles peut être infini mais dont les probabilités de passage s'expriment à l'aide de densités. Il obtient plusieurs résultats concernant ces densités à l'aide du théorème ergodique uniforme de Yosida et Kakutani [Voir aussi un article de G. Marinescu et du rapporteur, Ann. of Math. (2) **52** (1950), 140–147; MR **12**, 266]. Il définit, comme dans le cas dénombrable, les chaînes du type (*B*) et démontre un résultat correspondant au théorème (I).

C. T. Ionescu Tulcea (New Haven, Conn.)

2035:

Onicescu, O. La répartition limite des sommes de variables aléatoires d'un processus Markov fini, homogène et continu. An. Univ. "C. I. Parhon" Bucureşti. Ser. Şti. Nat. **5** (1956), no. 12, 9–12. (Romanian. French and Russian summaries)

In this paper the author gives various results concerning the homogeneous Markoff chains with a finite number of states and continuous parameter. In particular, he shows that the method introduced by himself and by G. Mihoc [Mathematica Cluj **16** (1940), 13–43; MR **1**, 342] for the study of the asymptotic properties of the partial sums of a sequence of random variables constituting a Markoff chain with a finite number of states may be adapted to the continuous case. Estimations similar to those obtained in the discrete case are also given.

C. T. Ionescu Tulcea (New Haven, Conn.)

2036:

Ciucu, G. Un cas d'exception pour la répartition limite des sommes de variables aléatoires d'un processus Markoff fini, homogéne et continu. Gaz. Mat. Fiz. Ser. A. **3** (1958), 141–143. (Romanian. French and Russian summaries)

This paper has as its subject the study of the homogeneous Markoff chains with a continuous parameter and finite number of states. It is a continuation of the one reviewed above. Various cases not covered by the analysis in that paper are treated here.

C. T. Ionescu Tulcea (New Haven, Conn.)

2037:

Nomoto, Hisao. Continuity of strong Markov processes. Sûgaku **9** (1957/58), 15–16. (Japanese)

The author gives an affirmative answer to a problem proposed by K. Ito: "Is it true that almost all the sample functions $X_t(\omega)$ of a strong Markov process X_t $(0\leqq t<\infty)$ have right-limit and left-limit for all $t\geqq 0$?". The author's proof relies upon the martingale theory. It is remarked at the end of the paper that, according to K. Ito, another proof was obtained by H. P. McKean by combining the martingale theory with the semi-group theory.

K. Yosida (Tokyo)

2038:

Gartšteĭn, B. N. On the limit-distribution of the maximal term for a case of homogeneous Markov chains. Užgorod. Gos. Univ. Naučn. Zap. **18** (1957), 195–200. (Russian)

2039:

Radu, N. C. Une propriété asymptotique des densités d'une chaîne de Markov continue au cours du temps. Com. Acad. R. P. Romîne **7** (1957), 929–932. (Romanian. Russian and French summaries)

Let $\phi(t, x; s, y)$ be the probability (density relative to a measure function μ on the state space) of a transition from x at time t to y at time s for a Markov process. Suppose that $\{t_n, n\geqq 1\}$ is a monotone sequence with limit ∞, that $\{\alpha(t_n), n\geqq 1\}$ is a sequence of positive numbers with sum ∞, and that M is a set of positive μ-measure such that (t fixed, $<t_1$)

$$\phi(t, x; t_j, z)\geqq\alpha(t_j) \text{ if } t<t_j, z\in M.$$

Then it is shown, following the classical reasoning of Markov, that, when $s\to\infty$,

$$|\phi(t, x; s, z)-\phi(t, y; s, z)|\to 0$$

uniformly as x, y, z vary. *J. L. Doob* (Urbana, Ill.)

2040:

Sankaranarayanan, G. Some asymptotic properties of Poisson process. Tôhoku Math. J. (2) **10** (1958), 60–68.

Let $X(t, \omega)$ be a Poisson process with parameter λ. Let $t_m(\omega)=\text{Min}[T, X(T, \omega)=m]$, $L_m(\omega)=t_{m+1}(\omega)-t_m(\omega)$, and $y_j=(L_{(j-1)n}+\cdots+L_{jn-1})/n$. Then $y_1, y_2, \cdots$ are independent random variables with the common gamma distribution $p(x)=((n\lambda)^n/\Gamma(n))x^{n-1}e^{-n\lambda x}$ if $x>0$. Let $u_m=\text{Max}(y_1, y_2, \cdots, y_m)$ and $e_m=\text{Min}(y_1, y_2, \cdots, y_m)$. Then it is proved that, for $0<\alpha<1$,

$$\Pr\left\{\liminf_{m\to\infty}\frac{n\lambda u_m}{\alpha\log m}\geqq 1\right\}=1,$$

and, for $\beta>1$,

$$\Pr\left\{\limsup_{m\to\infty}\frac{l_m}{\beta[(n!/(n\lambda)^n)(\log m)/m]^{1/m}}\leqq 1\right\}=1.$$

For $n=1$, these results were proved by T. Hida [Nagoya Math. J. **6** (1953), 29–36; MR **15**, 444]. The author also proves that $(t_1+t_2+\cdots+t_m)/m^2$ converges in probability to $1/2\lambda$. *J. L. Snell* (Palo Alto, Calif.)

2041:

Nisio, Makiko. Note on random Riemann sum. J. Math. Soc. Japan **9** (1957), 448–451.

Let t_i^n denote the ith jumping point of a Poisson process with parameter n. Theorem 1: If $f\in L_1(0, \infty)$, then $S_n=\sum_{i=1}^{\infty} f(t_i^n)(t_{i+1}^n-t_i^n)$ converges to $I=\int_0^\infty f(t)dt$ in probability as $n\to\infty$. Theorem 2: If $f\in L_1(0, \infty)\cap L_2(0, \infty)$, then S_{2^n} converges to I with probability one as $n\to\infty$. (It makes no difference whether the Poisson processes are independent or dependent.) The corresponding problem with t_i^n the ith smallest of n independent random variables uniformly distributed in (0, 1) has been treated by S. Takahashi [Tôhoku Math. J. (2) **8** (1956), 245–257; MR **19**, 399].

W. Hoeffding (Stanford, Calif.)

2042:

Hájek, Jaroslav. Predicting a stationary process when the correlation function is convex. Czechoslovak Math. J. **8**(83) (1958), 150–154. (Russian summary)

Let x_t, $-\infty<t<\infty$ be a wide sense stationary process with known mean μ, variance σ^2 and correlation function R_τ. Consider the linear predictor

$$\text{Pred } x_{t+\Delta}=\mu+\sum_{i=1}^{n} c_i(x_{t_i}-\mu),\ t_1<\cdots<t_n=t<t+\Delta,\ n\geqq 1.$$

The author proves that if R_τ is convex then the residual variance for this predictor satisfies the inequality

$$D\{x_{t+\Delta}-\text{Pred } x_{t+\Delta}\}\geqq\sigma^2(1-R_\Delta).$$

Since the variance for the best linear prediction based only on x_t is $\sigma^2(1-R_\Delta^2)$ this shows that in the case of a convex correlation function the relative reduction of the

residual variance attainable by making use of any number of preceding observations, cannot exceed 50%.

J. L. Snell (Palo Alto, Calif.)

2043:

Watanabe, Tsuyoshi. A remark on an additive process. Sûgaku **8** (1956/57), 215–216. (Japanese)

A simple proof is given to the following theorem: A temporally homogeneous additive process $x(t, \omega)$ on a real x-axis is t-continuous in probability if $x(t, \omega)$ is (t, ω)-measurable. This theorem was stated without proof and used in a paper by K. Ito [Trans. Amer. Math. Soc. **81** (1956) 253–263; MR **17**, 980].

K. Yosida (Tokyo)

2044:

***Doob, J. L. Interrelations between Brownian motion and potential theory.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 202–204. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Expository paper.

2045:

Watanabe, Yoshikatsu. Einige Erweiterung des Pólyaschen Irrfahrtproblem. J. Gakugei Tokushima Univ. **8** (1957), 13–25.

The author first corrects a result given in a previous paper [same J. **6** (1955), 41–49; MR **17**, 1097], namely that a point moving randomly on two plane lattices joined by risers at each lattice point behaves like a point on a space lattice; it is now shown to return to the origin infinitely often with probability 1, as on a plane lattice. Next, if the risers are all one way, the probability of return to the origin (in the plane from which the risers leave) is less than unity; indeed, in the author's picturesque phrase, this plane after a sufficiently long time becomes an uninhabited ruin. Next, the same type of behavior is found for a plane (x, y) lattice with one way movements in the y direction. Finally the plane lattice is reconsidered with probability p_0 of no movement, probability p_1 of moving to each of the four neighbors $(x\pm1, y)$ $(x, y\pm1)$ and probability p_2 of moving to each of the four (second degree) neighbors $(x\pm1, y\pm1)$, $(x\pm1, y\pm1)$; the behavior is similar to the regular plane lattice. Also remarks are made on the corresponding consideration of random flight with unit steps; in the plane the point returns to a point within a unit circle about the origin infinitely often, while for higher dimensions it is certain to wander off to infinity (as for random walks).

J. Riordan (New York, N.Y.)

2046:

Bellman, Richard; Kalaba, Robert; and Wing, G. Milton. On the principle of invariant imbedding and neutron transport theory. I. One-dimensional case. J. Math. Mech. **7** (1958), 149–162.

A homogeneous one-dimensional rod consisting of fissionable nuclei is considered in which neutrons are moving either to the right or to the left. The following processes are assumed to occur when neutrons collide with nuclei: 1. Production of two neutrons, the probability of such a fission taking place in an infinitesimal length Δ being $a\Delta$ (a=constant); 2. scattering of the neutron in the direction in which it has been travelling with a probability $b\Delta$ (b=constant); 3. scattering of the neutron in the direction opposite to which it has been travelling with a probability $c\Delta$ (c=constant); 4. disappearance of the neutron with a probability $f\Delta$ (f=constant). With these processes taking place, let $p_n(x)$ and $q_n(x)$ denote the probabilities that exactly n neutrons emerge over all time from the left, respectively the right, end of a homogeneous rod of length x as a result of one neutron entering at the left end at time zero. Considering the processes taking place in x, $x+\Delta$, we can express $p_n(x+\Delta)$ and $q_n(x+\Delta)$ in terms of $p_n(x)$ and $q_n(x)$, respectively, and passing to the limit $\Delta=0$ in the relations thus obtained, we find

$$(1)\quad p_n'(x)=-(n+1)hp_n(x)+\sum_{k=0}^{n} kp_k(x)[ap_{n-k}(x)+cp_{n-k+1}(x)]+c(n+1)p_{n+1}(x)p_0(x)+f(n+1)p_{n+1}(x)+ap_{n-1}(x)+c\delta_{1n},$$

$$(2)\quad q_n'(x)=-nhq_n(x)+a\sum_{k=0}^{n} kq_k(x)p_{n-k}(x)+c\sum_{k=0}^{n+1} kq_k(x)p_{n+1-k}(x)+(n+1)fq_{n+1}(x)\ (n>0),$$

where $h=a+c+f$. The equations governing the various moments of the distributions can be determined from the foregoing equations and can, in principle, be solved successively. Thus if

$$(3)\quad E(x)=\sum_{n=1}^{\infty} np_n(x),\quad S(x)=\sum_{n=2}^{\infty} n(n-1)p_n(x),$$

are the first and the second moments, it can be shown that

$$E(x)=\frac{E'(x)+(c+f)E(x)-(a+c)}{(a+c)E(x)-(c+f)},$$

$$(4)\quad S'(x)=2aE(x)+2aE^2(x)+3aE(x)S(x).$$

With the boundary conditions $E(0)=1$, equation (3) can be integrated explicitly.

The paper includes the results of computation of the functions $p_n(x)$ for $a=1$, $b=c=f=0$ for $n=1, 2, 3, 4, 5, 6, 10, 15$ and 20.

S. Chandrasekhar (Williams Bay, Wis.)

2047:

***Kac, Mark. Probability in classical physics.** Applied probability. Proceedings of Symposia in Applied Mathematics, Vol. VII, pp. 73–85. McGraw-Hill Book Co., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1957. $5.00.

The paper gives a brief review, from the point of view of the mathematician, of some problems connected with the application of probability considerations in classical physics. The problems are arranged into two main categories: equilibrium problems and nonequilibrium problems. Among the topics discussed are the Ising models, the problem of the imperfect gas, the Boltzmann equation and the "Stosszahlansatz". The unsolved basic difficulty of giving a mathematical justification of probabilistic reasonings in kinetic theory is briefly emphasized but not discussed.

H. Cramér (Stockholm)

2048:

Longuet-Higgins, M. S. The statistical distribution of the curvature of a random Gaussian surface. Proc. Cambridge Philos. Soc. **54** (1958), 439–453.

The probability distribution of the total curvature $\Omega=\kappa_a\kappa_b$ of a stationary random Gaussian surface is derived subject to the condition that the squares of the surface slopes are small compared to one. The distribution is specified by two parameters H and Δ, each expressible as combinations of the fourth moments of the spectrum of the displacement covariance. The distributions $p(\Omega)$ are shown to satisfy a third order linear differential equation; they are in general asymmetrical with a positive

skewness and a singularity at $\Omega=0$. The distributions are considered for some limiting conditions on the directional distribution of the displacement spectrum, including the cases of directionally peaked and isotropic spectra.

O. M. Phillips (Baltimore, Md.)

STATISTICS

2049:

***Graf, Ulrich; und Henning, Hans-Joachim. Formeln und Tabellen der mathematischen Statistik. Berichtigter Neudruck.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. vii+104 pp. DM 12.60.

A corrected new printing of the work (1953) reviewed in MR **15**, 140.

2050:

de Lucia, Luigi. Variabilità a due dimensioni. Fac. Sci. Statist. Demogr. Attuar. Ist. Statist. Ist. Calcolo Probab. Publ. no. 2 (1956), 13 pp.

The paper contains a discussion of several indices of variability for a two-dimensional random variable (X, Y) which, according to the author, were proposed by G. Pompilj. These indices are of the form $\{E[A^r]\}^{1/r}$, A being the area of the triangle with vertices (X_1, Y_1), (X_2, Y_2), (X_3, Y_3); where either all three vertices are independent observations of (X, Y), or two vertices are independent observations of (X, Y) and one is fixed, or one is an observation of (X, Y) and two are fixed.

Z. W. Birnbaum (Seattle, Wash.)

2051:

Dwass, Meyer. On several statistics related to empirical distribution functions. Ann. Math. Statist. **29** (1958), 188–191.

Let $X_1, \cdots, X_n$ be a sample of a random variable X with continuous c.d.f. $F(x)$, and $F_n(x)$ the corresponding empirical c.d.f. Consider the random variables: $U_n=\mu\{t:F_n(t)-F(t)>0\}$ where μ denotes Lebesgue measure; $D_n=\sup_t\{F_n(t)-F(t)\}$; $V_n=\inf_t\{F(t):F_n(t)-F(t)=D_n\}$. It is known that U_n and V_n have uniform probability distributions on (0, 1) [B. V. Gnedenko and V. S. Mihalevic, Dokl. Akad. Nauk SSSR (N.S.) **85** (1952), 25–27; MR **14**, 60; Z. W. Birnbaum and R. Pyke, Ann. Math. Statist. **29** (1958), 179–187; MR **20**#393]. The author gives a concise proof of these two statements, employing results of E. S. Andersen [Math. Scand. **1** (1953), 263–285; MR **15**, 444] and known properties of the Poisson process, and generalizes them to the case where $F_n(t)$ is replaced by a weighted mean of several empirical distribution functions corresponding to independent samples of the same random variable.

Z. W. Birnbaum (Seattle, Wash.)

2052:

Lindley, D. V. Fiducial distributions and Bayes' theorem. J. Roy. Statist. Soc. Ser. B **20** (1958), 102–107.

The note is concerned with conditions under which the fiducial distribution of a real parameter θ is identical with the posterior distribution of θ corresponding to a suitable prior distribution, and with conditions under which the fiducial distribution has certain consistency properties.

W. Hoeffding (Chapel Hill, N.C.)

2053:

Wallace, David L. Intersection region confidence procedures with an application to the location of the maximum in quadratic regression. Ann. Math. Statist. **29** (1958), 455–475.

Confidence region procedures for multidimensional quantities sometimes require prohibitive amounts of computation and the regions are difficult to represent in a useful way. Some approximate procedures are constructed by using regions obtained as the intersection of several regions, each much easier to construct. The procedures are applicable to the solution of simultaneous equations whose coefficients are subject to random error. Approximations by convex polyhedra and by parallelepipeds are proposed. The procedures are illustrated for setting a confidence region for the location of the vertex of a quadratic regression surface. (Author's summary.)

D. M. Sandelius (Göteborg)

2054:

Silverstone, H. Estimating the logistic curve. J. Amer. Statist. Assoc. **52** (1957), 567–577.

This paper considers the estimation of parameters of the logistic curve when the response at any given dose is binomial. In a comparison of the method of maximum likelihood and the method with "minimum logit chi-square" it is "shown that the maximum likelihood estimates, but not the minimum logit chi-square estimates, are sufficient estimators for the parameters. Particular attention is paid to samples where all but one of the responses are zero or 100 per cent. ...The two sets of estimators are also surveyed from the viewpoint of consistency and some remarks made on the criterion of minimum mean square error estimators when sufficient estimators exist." (From the author's summary.)

G. Kallianpur (East Lansing, Mich.)

2055:

***Blom, Gunnar. Statistical estimates and transformed beta-variables.** John Wiley and Sons, Inc., New York; Almqvist & Wiksells, Stockholm; 1958. 176 pp. $5.00.

The central objective of this study is the estimation of location and scale parameters by means of linear combinations of order statistics. Let z be a random variable with c.d.f. $F((z-\mu)/\sigma)$ where F is a known function while μ and σ are unknown parameters. Let $z_1, \cdots, z_n$ denote an ordered sample of n observations on z, and let $x_1, \cdots, x_n$ be the corresponding values of the reduced variable $x=(z-\mu)/\sigma$, with c.d.f. $F(x)$. Writing

$$(1) \qquad z_i=\mu+\sigma\xi_i+\sigma(x_i-\xi_i), \quad \xi_i=Ex_i,$$

one is confronted with a least-squares set-up where the means ξ_i and the error variances and covariances are determined by $F(x)$ and hence, in principle, known. The actual formation and solution of this set-up being in most cases impracticable, the author introduces a linear transformation of the set $z_1, \cdots, z_n$, chosen so as to make the covariance matrix easy to handle, at least asymptotically. Let λ_i be defined by $F(\lambda_i)=i/(n+1)$, and let

$$(2) \qquad C_{1i}=f(\lambda_i)-f(\lambda_{i+1}), \quad C_{2i}=f(\lambda_i)Ex_i-f(\lambda_{i+1})Ex_{i+1}.$$

Then, to find "nearly unbiased nearly best estimates" of μ and σ, one only has to calculate the square sums and the product sum of the quantities (2) and solve two linear equations. The means Ex_i appearing in (2) may often profitably be calculated from the approximation

$$(3) \qquad Ex_i \approx F^{-1}[(i-\alpha)/(n-\alpha-\beta+1)]$$

with appropriate constants α, β. — A study of the efficiency of the estimates thus obtained has led the author to an interesting modification of the Cramér-Rao inequality (in one and more dimensions). The modification is valid under milder conditions, and accounts for various well-known cases of hyper-efficiency. — A large part of the study is, of course, devoted to the estimation of remainder terms. *G. Elfving* (Helsingfors)

2056:

Silvey, S. D. The Lindisfarne scribes' problem. J. Roy. Statist. Soc. Ser. B **20** (1958), 93–101.

For deciding on the number of changes in the sequence $\pi_1, \cdots, \pi_s$ based on s independent binomial variables (n_i, π_i), procedures are constructed from the intersections of χ^2-tests. [For the general problem and recent history of the construction of multiple decision procedures from tests, see E. L. Lehmann, Ann. Math. Statist. **28** (1957), 1–25; MR **18**, 955.] *J. Hannan* (East Lansing, Mich.)

2057:

Hoeffding, Wassily; and Wolfowitz, J. Distinguishability of sets of distributions. (The case of independent and identically distributed chance variables). Ann. Math. Statist. **29** (1958), 700–718.

X_i are independently distributed vectors with common d.f. F. Two subsets $\mathscr{G}$ and $\mathscr{H}$ of a set $\mathscr{F}$ of distributions are said to be distinguishable ($\mathscr{T}$) if, within the class of tests $\mathscr{T}$, there is a test for which the maximum error probability for testing $F \in \mathscr{G}$ against $F \in \mathscr{H}$ is arbitrarily small for $F \in \mathscr{G} \cup \mathscr{H}$. Typical classes $\mathscr{T}$ which are considered are the classes of sequential tests for which, for $F \in \mathscr{F}$, the probability of termination is unity, the rth moment of the sample size N is finite, or N is bounded. Under various conditions on $\mathscr{F}$, corresponding conditions are proved to be necessary or sufficient (or both) for distinguishability. For example, for distinguishability relative to the first class above, if $d(F, G)=\sup_A|F(A)-G(A)|$ where A ranges over the Borel sets, then $\max[d(F, \mathscr{G}), d(F, \mathscr{H})]>0$ for $F \in \mathscr{F}$ is necessary and sufficient when $\mathscr{F}$ is almost any commonly encountered parametric family. The paper in part extends results of Kraft [Univ. of Calif. Publ. Statist. **2** (1955), 125–141; MR **17**, 505] and Hoeffding [Proc. Third Berkeley Symp. Math. Statist., Univ. of Calif. Press, Berkeley, 1956, pp. 105–116; MR **18**, 947]. *J. Kiefer* (Oxford)

2058:

Sarhan, A. E.; and Greenberg, B. G. Estimation of location and scale parameters by order statistics from singly and doubly censored samples. II. Tables for the normal distribution for samples of size $11 \leqq n \leqq 15$. Ann. Math. Statist. **29** (1958), 79–105.

In an earlier paper [Ann. Math. Statist. **27** (1956), 427–451; MR **18**, 238], the authors gave best linear systematic estimates of the mean and standard deviation of a normal distribution from singly and doubly censored samples of size $n \leqq 10$. In the present paper, the tables are extended to the case where $11 \leqq n \leqq 15$.

Benjamin Epstein (Stanford, Calif.)

2059:

Lukacs, Eugene. Certains tests indépendants de la distribution initiale. Ann. Inst. H. Poincaré **15** (1957), 252–265.

Let Ω denote a family of probability distributions, and $X_1, \cdots, X_n$ a sample of a random variable with distribution function $F(x) \in \Omega$. An expression $S(X_1, \cdots, X_n|F)$ which is a functional of $F \in \Omega$ and, for each F a measurable function in $X_1, \cdots, X_n$ with respect to the measure induced by F, is called a "statistic" on Ω. A statistic S is called "distribution-free" on Ω if the probability distribution of S is the same for all $F \in \Omega$. If S is of the form $S(X_1, \cdots, X_n)$, i.e., its value depends only on the sample but not on F, then it is called a "partition-statistic". — The first part of the paper deals with the family N of all normal distributions, its subfamilies N_μ having fixed mean μ, and its subfamilies N^σ having fixed variance σ^2. The following theorems are proven: a partition-statistic on N which has finite expectation $E(S|F)$ for all $F \in N$ is distribution-free on $N^\sigma \Leftrightarrow E(S|\bar{X})=E(S)$; it is distribution-free on $N_\mu \Leftrightarrow E(\sum_{j=1}^n (X-\mu)^2|S)=E(\sum_{j=1}^n (X_j-\mu)^2)$; it is distribution-free on $N \Leftrightarrow E(\sum_{j=1}^n X_j|S)=E(X_j)$ and $E(\sum_{j=1}^n X_j^2|S)=E(\sum_{j=1}^n X_j^2)$. The existence of nontrivial distribution-free statistics of these kinds is assured by the theorem: if L is a family of absolutely continuous probability distribution functions depending only on a location parameter μ and a scale parameter σ, then a partition-statistic S is distribution-free on $L \Leftrightarrow S(X_1, \cdots, X_n)$, $S(aX_1+b, \cdots, aX_n+b)$ have the same distribution for every $a \neq 0$ and b. — In the second part of the paper a similar study is made of partition statistics which are distribution-free on families of distributions of finite rank, as defined by E. B. Dynkin [Dokl. Akad. Nauk SSSR (N.S.) **75** (1950), 161–164; MR **12**, 427]. *Z. W. Birnbaum* (Seattle, Wash.)

2060:

Gupta, Shanti S.; and Sobel, Milton. On the distribution of a statistic based on ordered uniform chance variables. Ann. Math. Statist. **29** (1958), 274–281.

A statistic of the kind considered earlier by Epstein and Sobel [J. Amer. Statist. Assoc. **48** (1953), 486–502; Ann. Math. Statist. **25** (1954), 373–381; MR **15**, 143, 810] in their work on life testing when the underlying distribution of life is exponential, is now studied in the case where the underlying distribution of life is uniform over the unit interval. The exact distribution of the statistic is derived and asymptotic normality, under appropriate conditions, is derived.

Benjamin Epstein (Stanford, Calif.)

2061:

Sukhatme, Balkrishna V. Testing the hypothesis that two populations differ only in location. Ann. Math. Statist. **29** (1958), 60–78.

Let $X_1, X_2, \cdots, X_n$ be n independent identically distributed random variables with cumulative distribution function $F(x-\xi)$. Let $\hat{\xi}(X_1, X_2, \cdots, X_n)$ be an estimate of ξ such that $n^{\frac{1}{2}}(\hat{\xi}-\xi)$ is bounded in probability. The first part of this paper (Secs. 2 through 4) is concerned with the asymptotic behavior of U-statistics modified by centering the observations at $\hat{\xi}$. A set of necessary and sufficient conditions are given under which tne modified U-statistics have the same asymptotic normal distribution as the original U-statistics. These results are extended to generalized U-statistics and to functions of several generalized U-statistics. The second part gives an application of the asymptotic theory developed earlier to the problem of testing the hypothesis that two populations differ only in location. (Author's summary.) [For U-statistics see Hoeffding, Ann. Math. Statist. **19** (1948), 293–325; MR **10**, 134].

D. M. Sandelius (Göteborg)

2062:
Thompson, W. A., Jr. A note on P.B.I.B. design matrices. Ann. Math. Statist. **29** (1958), 919–922.

Bose's C-matrix for a PBIB design with m classes is written as a linear combination of the $m+1$ commutative and linearly independent matrices $I, B_1, \cdots, B_m$, where the (i, j)th element of B_k is 1 if $i \neq j$ and treatments i and j are k-associates, and is 0 otherwise.

J. Kiefer (Oxford)

2063:
***Cox, D. R. Planning of experiments.** A Wiley Publication in Applied Statistics. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1958. vii+308 pp. $7.50.

This book is addressed to readers with little mathematical background. Its purpose is to describe the situations which make the use of various experimental designs necessary or advisable and to explain the reasons for the appropriate analysis in terms that appeal to the intuition. Several topics that are not given prominence in other books are treated with care. The book contains for instance a careful exposition of the use of concomitant variables in reducing the experimental error, a topic to which the author himself has made a significant contribution. The fitting of response surfaces is also given an introductory treatment. The use of each design is explained by accounts of numerous experimental situations in which its use was found advisable.

This reviewer, being fully acquainted with the mathematical models which the author describes in non-mathematical terms, is not in a position to judge how much a reader who is not acquainted with these models can profit from the author's discussion. In reading the book, however, one often wishes that research workers in all fields could be prevailed upon to acquire the fairly limited mathematical background that would enable them to understand a presentation of a topic in statistics which neither seeks out nor avoids mathematical difficulties. *H. B. Mann* (Columbus, Ohio)

2064:
Chakravarti, I. M. Simplified proofs of some results in the theory of optimal designs. Sankhyā **19** (1958), 189–194.

In the usual linear hypothesis set up $EY=AP$, given the values of a set of diagonal elements of $A'A$, the minimum of the determinant or of the maximum eigenvalue of the covariance matrix of the corresponding set of best linear estimators is determined. [See also S. Ehrenfeld, Ann. Math. Statist. **26** (1955), 247–255; MR **17**, 56.] A condition for orthogonality of two estimators is given.

J. Kiefer (Oxford)

2065:
Bahadur, R. R. A note on the fundamental identity of sequential analysis. Ann. Math. Statist. **29** (1958), 534–543.

Assuming that $P_F\{n<\infty\}=1$ and that $\phi(t)=E_F e^{tz_1}<\infty$ for t in a neighborhood of the origin, the fundamental identity is generalized to $E_F\{\phi^{-n}(t)e^{tS_n}\}=P_{G_t}\{n<\infty\}$, where $s_n=X_1+\cdots+X_n$ and $dG_t(x)=[\phi(t)]^{-1}e^{tx}dF(x)$. Conditions which imply $P_{G_t}\{n<\infty\}=1$ and the differentiability under the expectation sign, are given. Bounds are given on the probability of absorption for a random walk with drift away from a single absorbing barrier.

J. Kiefer (Oxford)

2066:
Birnbaum, Allan. Sequential tests for variance ratios and components of variance. Ann. Math. Statist. **29** (1958), 504–514.

Let the potential observations $x_1, \cdots$ and $y_1, \cdots$ be normally and independently distributed with means zero and standard deviations σ_x, σ_y respectively. The sampling rule proposed runs as follows (g is a constant depending on the test problem at hand): At each stage, look at the partial sums of the sequences $gx_1^2, \cdots$ and $y_1^2, \cdots$, taken as far as the x's and y's are observed, and add a new observation of the variable exhibiting the smaller partial sum. The distribution theory is based on the fact that the sums $x_1^2+x_2^2$, $x_3^2+x_4^2$, etc., are exponentially distributed; hence, the even ones of the partial sums mentioned above constitute Poisson processes. — Efficiency questions are investigated; the sequential test is found to provide savings.

G. Elfving (Helsingfors)

2067:
Rajski, C. The life-tests distribution functions. Zastos. Mat. **3** (1958), 323–328. (Polish. Russian and English summaries)

In the theory of life-tests it is commonly assumed that the probability distribution function of the length of the life is an exponential one. The present paper is concerned with a search for a wider class of distribution functions which can reasonably be used as life distribution functions. Under very general assumptions the following theorem is deduced: $F(x)$ is a life distribution function if $[1-F(x)]^{-1}dF(x)/dx$ is a distribution function of any kind.

Further it is proved that the above condition is fulfilled by the distribution function of the gamma type, defined in the following manner:

$$F(x)=\begin{cases}0, & \text{if } 0\leqq x<c,\\ 1-e^{-(x-c)}\sum_{j=0}^{r}(x-c)^j/j!, & \text{if } c\leqq x<\infty,\end{cases}$$

with r natural and $c\geqq 0$, except for the case when $r=0$ and $c=0$. *Author's summary*

2068:
Parzen, Emanuel. On choosing an estimate of the spectral density function of a stationary time series. Ann. Math. Statist. **28** (1957), 921–932.

Let $x(t)$ be a continuous parameter stationary process with spectral density $f(w)$. The author studies estimates of $f(w)$ of the type

$$f_T^*(w)=(2\pi)^{-1}\int_{-T}^{T}e^{-ivw}k(B_Tv)R_T(v)dv,$$

where $k(u)$ is called the covariance averaging kernel, B_T tends to zero as T tends to infinity and $R_T(v)$ is the empirical covariance function. In terms of these quantities one can write down expressions for the asymptotic variance and bias of $f_T^*(w)$, valid under certain regularity assumptions.

To simplify the problem the author assumes that it is practically legitimate to deal with these expressions as if T were finite. This makes it possible to study the large sample mean square error as a function of T, B_T, functional form of $k(u)$ etc. This is done in some detail for certain families of $k(u)$ that contain most of the estimates suggested before. In a numerical investigation a very interesting conclusion is reached: the functional form of $k(u)$ is not as important as may have been thought at first glance. The practical consequences of this are discussed; to the reviewer it seems to indicate that the

bandwidth is the most important parameter to consider when choosing spectral estimates.

U. Grenander (Stockholm)

2069:

Fleischer, Isidore; and Kooharian, Anthony. On the statistical treatment of stochastic processes. Ann. Math. Statist. **29** (1958), 544–549.

The authors study statistical problems for stochastic processes and develop the approach of the reviewer [Ark. Mat. **1** (1950), 195–277; MR **12**, 511]. The basic tool is the Karhunen representation

$$x(t)=\sum_{k=1}^{\infty}\lambda_k^{\frac{1}{2}}z_k\varphi_k(t)$$

of the second order process $x(t)$ in terms of the eigen values λ_k and eigen functions $\varphi_k(t)$ of the corresponding covariance function. It is proved that the above representation converges weakly (on the probability space) in the mean (on the set of t-values). This result is used to derive Neyman-Pearson tests and maximum likelihood estimates by applying the martingale convergence theorem. As an illustration two periodic covariance functions are tested against each other. *U. Grenander* (Stockholm)

2070:

Siddiqui, M. M. On the inversion of the sample covariance matrix in a stationary autoregressive process. Ann. Math. Statist. **29** (1958), 585–588.

Let x_t be a stationary autoregressive process of order k and form its covariance matrix M_n of size $n\times n$. M_n is a Toeplitz matrix of special type and the author studies the inverse M_n^{-1}. For general Toeplitz matrices it is known how to obtain the inverse approximately, but the author shows that in the present case this can be done exactly in closed form if $n>2k$. The proof, which is simple and ingenious, is based on the observation that M_n^{-1} is the matrix appearing in the exponent of the frequency function of the normal process with the same moments; this matrix can then be obtained readily.

U. Grenander (Stockholm)

2071:

†van Dantzig, D. Mathematical problems raised by the flood disaster 1953. Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, Vol. 1, pp. 218–239. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam; 1957. 582 pp. $7.00.

A general account and bibliography. The problems fall into 3 categories: (1) statistical extrapolation problems, (2) econometric decision problems, (3) hydrodynamical problems. *F. Ursell* (Cambridge, England)

NUMERICAL METHODS

See also 1841, 2093, 2238.

2072:

Schröder, Johann. Über das Newtonsche Verfahren. Arch. Rational Mech. Anal. **1** (1957), 154–180.

This paper formulates iteration processes in a very general fashion and only the idea can be sketched in a brief review. Let $\mathfrak{R}$ be a semi-ordered linear space; then $\mathfrak{R}$ is an $\mathfrak{R}$-metric space if there is a function δ on $\mathfrak{R}\times\mathfrak{R}$ to $\mathfrak{R}$ which is definite, symmetric, and satisfies the triangle inequality. Also, $\mathfrak{R}$ is an $\mathfrak{R}$-normed space if there is an $\mathfrak{R}$-valued normlike function on $\mathfrak{R}$. First it is assumed that $\mathfrak{R}$ is a complete $\mathfrak{R}$-metric space and that $\mathfrak{F}\subset\mathfrak{D}\subset\mathfrak{R}$. Let T be a function on $\mathfrak{D}\times\mathfrak{F}$ to $\mathfrak{R}$ and consider the iterations of the form

$$u_{n+1}=T(u_n, x_n)\ (n=0, 1, 2, \cdots).$$

It is supposed that T is majorized in an appropriate sense by a function $\bar{T}$ in $\mathfrak{R}$ and that a host of Lipschitz conditions are in force. It is seen that the convergence of a related majorizing iteration for $\bar{T}$ implies the convergence for T. In the case that $\mathfrak{R}$ is a complete $\mathfrak{R}$-normed space, Newton's method is considered in a similar manner. An application is given to the calculation of an eigenvalue and a corresponding eigenvector of a linear operator in a Banach space, the special case of a matrix also being considered. *R. G. Bartle* (Urbana, Ill.)

2073:

Akušskiĭ, I. Ya. On the solvability of an inhomogeneous operation cycle. Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh. **1** (1958), 111–125. (Russian)

2074:

Altman, M. Über die Gausssche Methode der kleinsten Quadrate. Math. Nachr. **17** (1958), 9–15.

For solving the least-squares problem $Ax=y+d$ with $d^Td=\min$, the author first considers orthogonalizing the columns of A in sequence; then shows analytically that the sequence obtained by means of cyclic projection of a residual upon the column vectors approaches the solution in the limit. {These techniques are well known, and have been discussed by this reviewer in particular [J. Soc. Indust. Appl. Math. **3** (1955), 67–72; MR **17**, 536].}

A. S. Householder (Oak Ridge, Tenn.)

2075:

Popoviciu, Tiberiu. Sur la précision du calcul numérique dans l'interpolation par le polynome Newton à noeuds équidistantes. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Şti. Ser. I. **6** (1955), no. 3–4, 27–35. (Romanian. Russian and French summaries)

In calculating the value of Newton's polynomial,

$$f(a+xh)\approx L(x)\equiv\sum_{\nu=0}^{n}\frac{x(x-1)\cdots(x-\nu+1)}{\nu!}\Delta_h{}^{\nu}f(a),$$

one uses the algorithm

$$y_{\nu+1}=\Delta_h{}^{n-\nu}f(a)+\frac{x-n+\nu}{n-\nu+1}y_\nu,\ \nu=0, 1, \cdots, n;$$

$$y_0=0;\ y_{n+1}=L(x)$$

Approximations used in the successive calculations result in an accumulation of errors in the final value y_{n+1}. The author develops a formula for the total error c, and thereby obtains a bound on $|c|$, in terms of x and of the maximum absolute value of the successive errors. For discrete values of x, by tenths from 0 to 1, tables of factors are constructed to facilitate the calculations.

E. F. Beckenbach (Los Angeles, Calif.)

2076:

Feix, Marc; Sajaloli, Cécile; et Kuntzmann, Jean. Une variante de la méthode de Tricomi-Picone pour l'inversion de la transformation de Carson. Chiffres **1** (1958), 63–74.

In the Tricomi-Picone method for the computation of $f(x)$ knowing its Laplace transform $F(p)$ for real p, the accuracy obtained depends very much on the choice of a parameter s, $0<s<1$, since — as the authors observed — the order of magnitude of the coefficients $b_{Nj}(s)$ in

$$f(x)\approx f_N(x)=\sum_{j=0}^{N}b_{Nj}(s)\cdot F[-x^{-1}(j+1)\log s]$$

fluctuates violently, when s varies in (0, 1).

Large values of $b_{Nj}(s)$ are not useful since they amplify too much the errors affecting the values of F in the righthand member. The authors propose to use $s=s_0$, where s_0 minimizes the sum S_N of squares of coefficients b_{Nj}. They recommend using the values $e^{-0.63866}$, $e^{-1.04982}$ and $e^{-0.65971}$ for $N=3$, 4 and 5 respectively.

E. Kogbetliantz (New York, N.Y.)

2077:

Catton, Diana; and Millis, B. G. Numerical evaluation of the integral $(2\pi)^{-1}\int_{-\infty}^{+\infty}(\lambda\alpha^3+\alpha^2-1)^{-1}e^{-\alpha\omega}d\alpha$. Proc. Cambridge Philos. Soc. **54** (1958), 454–462.

For $0<\lambda\leq 2/(3\sqrt{3})$, ω real, the above integral is a solution to the equation

$$2\lambda\omega y'''+(3\lambda-2\omega)y''-2y'+2\omega y=0,$$

where the primes denote differentiation with respect to ω. Tables of the integral are given for $\frac{1}{2}\cdot 3^{\frac{1}{2}}\lambda=0(0.1)1$ to six decimals when $|\omega|=0(0.1)2.0(0.2)4.8$ and to seven decimals when $|\omega|=5(0.5)10(1)15$.

O. M. Phillips (Baltimore, Md.)

2078:

Sabroff, Richard R.; and Higgins, T. J. A critical study of Kron's method of "tearing". V. Matrix Tensor Quart. **8** (1958), 106–112.

2079:

Thacher, Henry C., Jr. Generalization of concepts related to linear dependence. J. Soc. Indust. Appl. Math. **6** (1958), 288–299.

A point $y=(y^{(0)}, y^{(1)}, \cdots, y^{(s)})$ in the $(s+1)$-dimensional space $S_1=S_1(s+1)$ is said to be a "degree-n combination" of p such points $x_1, \cdots, x_p$ if a set of p constants a_j exists such that $\prod_{k=0}^{s}(y^{(k)})^{n_k}=\sum_{j=1}^{p}a_j\prod_{k=0}^{s}(x_j^{(k)})^{n_k}$ for every system of positive integers n_k that satisfies the condition $(+)$ $\sum_{k=0}^{s}n_k=n$. In these terms a "degree-n dependence" and independence is defined, as well as a degree-n basis for a collection of points in S_1. If $W=W_n(s)$ is the number of terms of a homogeneous polynomial of degree n in the $s+1$ variables $x^{(0)}, x^{(1)}, \cdots, x^{(s)}$ with indeterminate coefficients (i.e. the number of sets of n_k satisfying $(+)$) then with the point x in S_1 with these coordinates is associated a certain point ξ in the W-dimensional space $S_n(s+1)$ whose coordinates are the terms of the polynomial deprived of the coefficients; thus if $n=2$, $s=2$ one has $x=(x^{(0)}, x^{(1)}, x^{(2)})$ and $\xi=(x^{(0)2}, x^{(0)}x^{(1)}, x^{(0)}x^{(2)}, x^{(1)2}, x^{(1)}x^{(2)}, x^{(2)2})$. The theory of these concepts is reduced to the theory of the corresponding linear concepts by means of the following theorem: A set of points x in S_1 is degree-n dependent if and only if their images ξ in S_n are linearly dependent. After this all facts concerning linear dependence and independence have their images in the generalized concepts presented in this paper. {Reviewer's remark: Although hardly ever given in this form before they should be contained in theorems of tensor algebra.} The applications are concerned with homogeneous polynomials of degree n with $W-1$ degree-n independent roots $x_1, \cdots, x_{W-1}$; these are proportional to the polynomial given by the determinant whose columns have the elements $\prod_{k=0}^{s}(x_\nu^{(k)})^{n_k}$ $(\nu=1, \cdots, W-1)$, one column for each set of n_k satisfying the condition $(+)$. The theorem in turn has applications to the author's quadrature formulae [cf. Math. Tables Aids Comput. **11** (1957), 189–194; MR **19**, 883].

H. Schwerdtfeger (Montreal, Q.)

2080:

Whaples, G. A note on degree-n independence. J. Soc. Indust. Appl. Math. **6** (1958), 300–301.

This note gives an answer to the following question suggested by the paper reviewed above: For what surfaces of degree n in $S_1(s)$ (i.e. in projective s-dimensional space) is the upper bound of $W_n(s)-1$ degree-n independent points actually attained? Further it is shown that if $s=2$, any five different points are degree-2 independent if and only if none of them is a scalar multiple of another and no four of them are collinear.

H. Schwerdtfeger (Montreal, Q.)

2081:

Krein, S. G.; and Prozorovskaya, O. I. An analogue of Seidel's method for operator equations. Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. **5** (1957), 35–38. (Russian)

An extension to Hilbert space of the theorem of Reich that if A is hermitian with positive diagonal, then for convergence of the Seidel process it is necessary, as well as sufficient, that A be positive definite. The authors seem unaware of Weissinger's proof of sufficiency [Z. Angew. Math. Mech. **33** (1953), 155–163; MR **15**, 66], although their proof is quite similar and implicitly utilizes an identity stated in more general form by Weissinger. This reviewer showed [Oak Ridge Nat. Lab. Rep., **ORNL-1883** (1955); MR **17**, 790] that for matrices the same identity implied necessity, and the proof is formally the same in more general spaces.

A. S. Householder (Oak Ridge, Tenn.)

2082:

Akušskiĭ, I. Ya. On the solvability of a problem in computation for a triangular matrix. Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh. **1** (1958), 126–132. (Russian)

2083:

Head, J. W.; and Oulton, G. M. The solution of "ill-conditioned" linear simultaneous equations. Aircraft Engineering **30** (1958), no. 356, 309–312.

The authors discuss the problem of solving a set of simultaneous equations where the matrix of coefficients is symmetric and 'ill-conditioned' in the sense that it has one (or two) very small latent roots. Only the projection of the solution vector in the direction of the latent vector corresponding to this small latent root is inaccurately determined by the simultaneous equations. They describe a method of obtaining all other components of the solution. Also discussed are two methods of evaluating the smallest latent root and its latent vector, these being required before the solution of the equations can be obtained.

B. Chartres (Providence, R.I.)

2084:

Henrici, Peter. On the speed of convergence of cyclic and quasicyclic Jacobi methods for computing eigenvalues of Hermitian matrices. J. Soc. Indust. Appl. Math. **6** (1958), 144–162.

The author considers the speed of convergence of a number of different Jacobi methods for diagonalizing Hermitian matrices. In particular, he discusses the classical Jacobi method, various cyclic methods, a restricted Jacobi method and a cyclic method with the thresholds as well as quasi-cyclic methods. He shows how to evaluate the "speed" of convergence of these procedures and derives a number of very interesting bounds.

H. H. Goldstine (Princeton, N.J.)

2085:

Bodewig, E. Zu Stiefels Berechnung der Eigenwerte aus den Schwarzschen Konstanten. Z. Angew. Math. Mech. **38** (1958), 72–73.

The paper is concerned with Rutishauser's quotient-difference algorithm for computing the eigenvalues of a symmetric matrix A from its Schwarzian constants [see Stiefel, same Z. **33** (1953), 260–262; MR **15**, 471; Rutishauser, Arch. Math. **5** (1954), 132–137; Z. Angew. Math. Physik **5** (1954), 233–251, 496–508; MR **16**, 176, 863]. The author wants to improve the algorithm in the case of multiple eigenvalues. He computes at first the multiplicity of an eigenvalue λ as root of the minimum polynomial and proposes a scalar deflation (applied to the Schwarzian constants) in order to eliminate λ and to proceed to the determination of the remaining eigenvalues. It must be observed however that Rutishauser's algorithm does all this pretty well, because it yields unambiguously the multiplicity of λ as root of the minimum polynomial and all the eigenvalues simultaneously.

Moreover Rutishauser has shown that the quotient-difference algorithm started from its first column (that is from the Schwarzian constants) is in general numerically unstable; therefore he introduced the "progressive form" of the algorithm beginning with the first diagonal. The reviewer is afraid that the same instability arises if the scalar deflation proposed by the author is used. The paper contains furthermore a method of matrix deflation for computing the multiplicity of an eigenvalue as root of the characteristic polynomial if A has only linear elementary divisors. *E. Stiefel* (Zürich)

2086:

Fisher, Michael E.; and Fuller, A. T. On the stabilization of matrices and the convergence of linear iterative processes. Proc. Cambridge Philos. Soc. **54** (1958), 417–425.

If the real matrix P of order n possesses a nested sequence of non-zero principal minors of all orders up to and including n, then there exists a diagonal matrix D such that the characteristic roots λ_i of DP are real and satisfy any one of the conditions: $\lambda_i>G$, $\lambda_0<-G$, $G>\lambda_i>0$, $0>\lambda_i>-G$, where G is an arbitrary positive scalar. This is stated as a corollary to the main theorem, but it is stronger than that theorem. Theorem 2, which follows from the "corollary", illustrates the others: For the same matrix P, a diagonal matrix D exists such that the total-step iteration $y_{k+1}=Df+(I-DP)y_k$ converges to the solution of $Py=f$.

The proof is constructive, but unfortunately the construction does not appear to be useful in practice.

A. S. Householder (Oak Ridge, Tenn.)

2087:

***Gravalos, F. G. A method of integrating the equations of motion of a body entering an arbitrary atmosphere with an automatic error analysis.** Proceedings of the VIIIth International Astronautical Congress, Barcelona 1957, pp. 156–164. Springer-Verlag, Vienna, 1958. vii +607 pp. $29.75.

A very simple method for the integration of ordinary differential equations is developed. The range of applicability is not restricted beyond the Cauchy-Lipschitz condition. Since the utilization of the method is dependent upon the use of digital, electronic computing machines, the manner of programming — and related questions — for an I.B.M. 704 is given.

The method is applied to the equations of motion of a body entering an arbitrary atmosphere and numerical examples are worked out. The salient aspect of this method is the continuous tracing of the error committed in each step. The author's view on the problem of error propagation as a disturbance is presented.

Author's summary

2088:

Crandall, Stephen H. Optimum recurrence formulas for a fourth order parabolic partial differential equation. J. Assoc. Comput. Mach. **4** (1957), 467–471.

This note concerns finite difference approximations to the equation $\partial^4\psi/\partial x^4+\partial^2\psi/\partial t^2=0$ in the strip $0\leq x\leq 1$, $t>0$. The author gives a general class of recurrence formulas for the approximation process, which includes as special cases previous formulas given by Collatz [Z. Angew. Math. Mech. **31** (1951), 392–393; MR **13**, 693], Royster and Conte [Proc. Amer. Math. Soc. **7** (1956), 742–749; MR **17**, 1214], Conte [J. Assoc. Comp. Mach. **4** (1957), 18–23; MR **19**, 1084], and the author himself [ibid. **1** (1954), 111–118; MR **16**, 525]. Other topics treated are truncation error, stability of the recurrence formulas, and rate of convergence for a particular set of boundary conditions. *J. Elliott* (New York, N.Y.)

2089:

Miller, G. F. A note on the numerical solution of certain non-linear integral equations. Proc. Roy. Soc. London. Ser. A **236** (1956), 529–534.

In der Arbeit werden die beiden nichtlinearen Integralgleichungen

$$x^2=\int_0^x \frac{f(y)dy}{[1+f(x)-f(y)]^{1/2}}, \tag{1}$$

$$x^2=\int_0^x \frac{[g(y)]^{\frac{1}{2}}dy}{[1+g(x)-g(y)]^{1/2}}, \tag{2}$$

$f(\infty)=0$, $g(\infty)=0$, betrachtet. Einige Aussagen über das Verhalten der Lösung von (1) fließen direkt aus der Integralgleichung. Die Lösung im Intervall $0\leq x\leq\frac{1}{2}$ wird durch Iteration gewonnen; im Bereich $x>\frac{1}{2}$ wird schrittweise eine numerische Integration vorgenommen. Die asymptotische Darstellung $f(x)\sim Ae^{-k^2x}$ ($A>0$ eine Konstante, $k^2=-\frac{1}{2}f''(\frac{1}{2})>0$) für große x wird angegeben. Entsprechende Aussagen gelten für $g(x)$. $f(x)$ und $g(x)$ sind für $0{,}00\leq x\leq 1{,}50$ (Schrittabstand 0,02) tabelliert.

G. Hämmerlin (Zbl **70**, 128)

2090:

Eremeev, N. V. A nomographic method for graphical differentiation. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 6, 3–6. (Russian)

2091:

Elbert, Donna D. Bessel and related functions which occur in hydromagnetics. Astrophys. J. Suppl. Ser. **28** (1957), 77–106.

Let

$$\mathfrak{J}_{n+1/2}(a_{j,s}r)=(\pi/[2a_{j,s}r])^{\frac{1}{2}}J_{n+1/2}(a_{j,s}r),$$

$$\mathfrak{F}_{n,j}=(n+1)(2n+3)^{-2}\{(a_{j,n}r)[nJ_{n+1/2}(a_{j,n}r)-(n+3)J_{n+3/2}(a_{j,n}r)]-n(2n+3)J_{n+3/2}(a_{j,n}r)\},$$

$$\mathfrak{G}_{n,j}=(n+2)(n+1)^{-1}\mathfrak{J}_{n,j}+(n+2)J_{n+3/2}(a_{j,n}r),$$

$$\mathfrak{L}_{n,j}=-(n+2)\{\mathfrak{F}_{n,j}+(n+1)(2n+3)^{-1}(2n+5)^{-1}(a_{j,n}r)^2J_{n+3/2}(a_{j,n}r)\},$$

$$\mathfrak{R}_{n,j}=(n+1)\{\mathfrak{G}_{n,j}-(n+2)(2n+1)^{-1}(2n+3)^{-1}(a_{j,n}r)^2J_{n+3/2}(a_{j,n}r)\},$$

where $a_{i,j}$ is the ith zero of $J_{j+1/2}(x)$. Tables are computed for the range 0.00(.01)1.00 in r and various combinations of the integers 0 through 4 for n, i, j for the functions $\mathfrak{J}_{n+1/2}(a_{i,j}r)$, $(\frac{1}{2})[\mathfrak{J}_{n-1/2}(a_{i,j}r)-\mathfrak{J}_{n+1/2}(a_{i,j}r)]$, $(\pi/[2a_{i,j}r])^{\frac{1}{2}}\mathfrak{F}_{n,j}$, $(\pi/[2a_{i,j}r])^{\frac{1}{2}}\mathfrak{G}_{n,j}$, $-(n+2)^{-1}(\pi/[2a_{i,j}r])^{\frac{1}{2}}\mathfrak{L}_{n,j}$, and finally $(n+1)^{-1}(\pi/[2a_{i,j}r])^{\frac{1}{2}}\mathfrak{R}_{n,j}$. *R. G. Langebartel* (Urbana, Ill.)

2092:

***Onoe, Morio. Tables of modified quotients of Bessel functions of the first kind for real and imaginary arguments.** Columbia University Press, New York, 1958. vii+338 pp. $12.50.

The functions tabulated are $\mathfrak{J}_n(z)=zJ_{n-1}(z)/J_n(z)$ for $z=x$ and $z=ix$, $x=0(0.01)20$, $n=1(1)16$. When the argument is real, values are generally given to 8S and when it is imaginary, 9S or 10S are given. In the introduction there is a collection of formulae relating to quotients of cylinder functions $C_\nu(z):\mathfrak{C}_\nu(z)=zC_{\nu-1}(z)/C_\nu(z)$ and $\mathfrak{C}_\nu(z)=zC_{\nu+1}/C_\nu(z)$. There is also a description of the computation of the tables. The power series for $J_{99}(z)$, which has radius of convergence $j_{99,1}>99$, was evaluated for the whole range of arguments and the tabulated values were obtained by a backward recurrence carrying 15 digits. This is straightforward in the imaginary case, but a special device has to be used to get over the poles in the real case. The values were checked by computing sixth differences at $n=16$ and $n=1$ in the imaginary case and at $n=16$ in the real case. The values for $n=1$, $n=16$ were spot-checked by separate computation of the functions at interval of 0.5 in x. It is stated that the error is less than half a unit in the last place and that 5-point Lagrangian interpolation will give full accuracy, except in the neighborhood of the poles.

Calculations were carried out on IBM equipment (602, 607, 650) and the table is printed by offset methods from copy prepared by an IBM 407; the reproduction is tolerable.

There is a short bibliography which indicates some of the areas in which quotients of cylinder functions have turned up. *J. Todd* (Pasadena, Calif.)

COMPUTING MACHINES

See also 2087, 2231.

2093:

***Grabbe, Eugene M.; Ramo, Simon; and Wooldridge, Dean E., (editors). Handbook of automation, computation, and control, Vol. 1: control fundamentals.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xx+999 pp. $17.00.

There are five general topics treated in this work: A. General Mathematics. B. Numerical Analysis. C. Operations Research. D. Information Theory and Transmission. E. Feedback Control. Under A, the following fields are discussed: sets; algebraic equations; matrices; difference, differential and integral equations; complex variables; operational mathematics; Laplace transforms; conformal mapping; Boolean algebra; probability; statistics. The selection of topics in section B is attuned throughout to the requirements of automatic computation. The whole work is written on a very high level of presentation and should be most useful to mathematicians interested in the fields under discussion.

2094:

***Richards, R. K. Digital computer components and circuits.** D. Van Nostrand Company, Inc., Princeton-Toronto-London-New York, 1957. vii+511 pp. $10.75.

This volume is a companion book to the author's "Arithmetic operations in digital computers" [Van Nostrand, 1955; MR 17, 672]. In that book the emphasis was on arithmetic in a computer and on 'logical', as distinct from 'engineering', design and organisation. The present book is intended to supply engineers with the information needed to reduce the ideas about arithmetic and logic to a working machine.

2095:

Arin', È. I.; and Šneps, M. A. Symbolic programming on the electronic computer of the Latvian Academy of Sciences. Latvijas PSR Zinātņu Akad. Vēstis **1958**, no. 6(131), 101–107. (Russian. Latvian summary)

The digital computer of the Latvian Academy of Sciences is a slightly modified version of the Soviet machine M-3, with a 30-bit word, a 1024-word internal memory unit, and no external memory devices.

This paper gives the sequence diagram (following Liapounov's technique described in Kitov's book "Elektronye Tsifrovye Mashiny") as well as a complete listing of the program in M-3 code, for a symbolic assembly program. The M-3 has a two-address instruction format, with an operation code of six bits, and two addresses of twelve bits each. The assembly program translates into this word structure from an external "symbolic" language consisting of sequences of octal digits. A symbolic address consists of six octal digits, the first two of which give a base address of a list. A Dewey-decimal-like address structure may be set up using the last four positions. Absolute addresses are distinguishable from such symbolic addresses by use of a code in the first two digits. An external, symbolic language sequence, representing a word's symbolic address, and the contents, an instruction or number, occupies 20 octal digits (two internal machine words). The entire system is similar to that of Rochester in Trans. IRE **EC-2** no. 1, 1953 (Professional group on electronic computers).

J. W. Carr, III (Ann Arbor, Mich.)

2096:

Beck, F. Harmonic analysis using a digital computer. Comput. J. **1** (1958), 117.

MECHANICS OF PARTICLES AND SYSTEMS

See also 1825, 1980, 2220, 2232.

2097:

***Ames, Joseph Sweetman; and Murnaghan, Francis D. Theoretical mechanics: an introduction to mathematical physics.** Dover Publications, Inc., New York, 1958. ix+462 pp. $2.00.

An unaltered reproduction of first edition, Ginn and Co., Boston, Mass., 1929.

2098:

Myller, A. Sur la construction d'une table. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), no. 1–2, 1–6. (Romanian. Russian and French summaries)

On expose la théorie mathématique de la construction

d'une table élliptique à trois pieds, la môde d'emplacement de ceux-ci et une formule de stabilité. *Résumé de l'auteur*

2099:

Arrighi, Gino. I fondamenti della statica in una trattazione logico-deduttiva. Boll. Un. Mat. Ital. (3) **12** (1957), 679–688.

E ben noto che una trattazione assiomatica della statica è stata più volte considerata nel lungo processo di evoluzione di questa dottrina. Sebbene nelle trattazioni moderne della meccanica la statica venga considerata come un caso particolare della dinamica, pure può essere interessante, in analogia allo sviluppo delle matematiche, specialmente della geometria, chiedersi se sià possibile sviluppare l'intera statica su di un minimo numero di postulati indipendenti. A questa domanda non si può dire che sià stata data una risposta completa, sebbene contributi recenti dovuti soprattutto a G. Giorgi e ad Hamel abbiano illuminato le difficoltà che si incontrano in tal genere di ricerche.

Lo scopo di questo lavoro è di approfondire il punto di vista di Andrade (della cosidetta "scuola del filo"), ripreso da G. Giorgi e di arrivare fino alla dimostrazione della regola del parallelogramma. *G. Lampariello* (Roma)

2100:

†Dobrowolski, W. W. Theorie der Mechanismen zur Konstruktion ebener Kurven. Akademie Verlag, Berlin, 1957. viii+134 pp. DM 18.00.

Übersetzung einer 1953 erschienenen russischen Ausgabe [Izdat. Akad. Nauk, Moscow]. Das Literaturverzeichnis nennt fast ausschliesslich russische Arbeiten. Ziel des Buches sind Mechanismen, womöglich von grosser Allgemeinheit anzugeben, womit man geometrisch oder analytisch definierte Kurven konstruieren kann. So wird ein "Universalkonikograph" von 14 Gliedern gegeben, das aus Brennpunkt, Leitlinie und Exzentrizität jeden Kegelschnitt erzeugen kann und das sowohl die Punkte wie die Tangenten liefert. Der Konstruktion der Kegelschnitte ist fast die Hälfte des Buches gewidmet und man begegnet dabei manchen einfachen und hübschen Sätzen; so wird z.B. gezeigt, dass im Viergelenk getriebene $ABCD$ mit den festen Punkten A und B die Senkrechte l von C auf BD einen Kegelschnitt umhüllt, der A als Brennpunkt hat und dass der Tangentialpunkt von l auf AD liegt. Daraus werden dann Konstruktionsmechanismen abgeleitet. Andere beruhen auf der projektiven Erzeugung des Kegelschnittes oder sie benutzen die Darstellung von projektiven und quadratischen Verwandtschaften. Das zweite Kapitel handelt über die Konstruktion von Kurven höherer Ordnung. Als Beispiele führen wir an: Lissajoussche Kurven, die Grassmannsche lineale Erzeugung, die Herstellung mehrdeutiger Zuordnungen durch Zahnradgetriebe, Inversoren, das Joukowski Profil. (Die theoretischen Betrachtungen sind dem Ref. nicht immer verständlich; auf S. 94 wird von einer Kurve ausgesagt, dass sie eine Gattung $p \neq 0$ hat, obwohl sie rational ist.) *O. Bottema* (Delft)

2101:

Mangeron, D.; et Drăgan, Corneliu. Sur une nouvelle méthode tensorielle dans la théorie des mécanismes. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), no. 1–2, 151–164. (Romanian. Russian and French summaries)

G. Kron [A short course in tensor analysis for electrical engineers, Wiley, New York, 1942; MR **4**, 29] has shown how tensors and transformation theory may be used to study many types of machinery. The authors of this paper use transformation theory for a similar study of mechanisms. They introduce projective coordinates in two Cartesian orthogonal frames attached to the mechanism in any two positions. By relating these coordinates, they determine the position transformation matrix, velocity transformation matrix, etc.

N. Coburn (Ann Arbor, Mich.)

2102:

Zil'berman, Ya. S. [Zilberman, I. S.] Application de la théorie des accélérations réduites à l'analyse cinématique des mécanismes plans de troisième classe. Bul. Inst. Politehn. Iaşi (N.S.) **3** (1957), no. 1–2, 165–174. (Russian. French and Romanian summaries)

"By use of a graphical-analytical method involving reduced accelerations due to D. Mangeron and C. Dragan [same Bul. **1** (1955), 229–249; **2** (1955), 295–304; MR **18**, 428], the author analyzes plane mechanisms of the third class. The method is completely geometrical. An example is worked out to illustrate the method. Finally, the author notes that one may apply this method to the determination of the centers of curvature of various points of a plane mechanism." (From French summary.)

N. Coburn (Ann Arbor, Mich.)

2103:

Sherwood, A. A. The mechanical generation of simple harmonic motion by three-dimensional linkages. Austral. J. Appl. Sci. **9** (1958), 96–104.

The paper describes a method for generating simple harmonic motion by means of a crank and slider mechanism. It is well known that exact simple harmonic motion of the slider does not result from uniform rotation of the crank, if the mechanism is arranged in two dimensions. The author shows, however, that by arranging the mechanism in three dimensions in a suitable manner exact simple harmonic motion can be obtained from uniform motion of the crank. Similar methods are applied to angular simple harmonic motion with less fortuitous results. *D. C. Lewis, Jr.* (Baltimore, Md.)

2104:

Nadile, Antonio. Sulle equazioni canoniche del moto dei sistemi anolonomi. Atti Accad. Peloritana Pericolanti. Cl. Sci. Fis. Mat. Nat. (3) **3(48)** (1945–49), 40–45 (1953).

The well-known connection between the Hamiltonian equations $\dot{q}_t=\partial H/\partial p_t$, $\dot{p}_t=-\partial H/\partial q_t$ and the Pfaffian $\sum_{t=1}^{n} p_t dq_t - H dt$ is extended to the more general case in which so-called quasi-coordinates are used. Since in a system having m $(<n)$ non-holonomic constraints, the quasi-coordinates $r_1, \cdots, r_n$ can be chosen so that the constraint equations appear in the simple form $dr_k=0$, $k=1, \cdots, m$, there is some formal advantage in using this type of generalization despite the far greater generalization appearing in connection with the Birkhoff-Pfaffian equations of dynamics. This is because of the fact that $2n-m$ of the original $2n$ equations of the unconstrained system remain valid when the system is modified by imposition of the m constraints, assuming that the system of quasi-coordinates is chosen as indicated above. *D. C. Lewis, Jr.* (Baltimore, Md.)

2105:

Grioli, Giuseppe. Movimenti dinamicamente possibili per un solido asimmetrico soggetto a forze di potenza nulla. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 459–463.

Determination of dynamically possible so-called "semi-

regular" precessional motions of an asymmetrical rigid body subjected to certain forces (like those of a magnetic field) which do no work. A precession is "semi-regular" if the precessional angular velocity is constant while the spin is not. *D. C. Lewis, Jr.* (Baltimore, Md.)

2106:

Išlinskiĭ, A. Yu. Theory of the double-gyroscopic vertical. Prikl. Mat. Meh. **21** (1957), 175–183. (Russian)

The matter in question in this paper is a gyroscopic system quite analogous to the author's previously described and theoretically explained "horizontal gyrocompass" [Prikl. Mat. Mech. **20** (1956), 487–499; MR **18**, 429]. The only difference between the previous and the new gyroscopic system is in the position of the centre of gravity of the gyroframe, which is the base of the whole construction. The theory is therefore given briefly. From this little change of the construction, the author expects a more precise determination of the vertical.

T. P. Anđelić (Belgrade)

2107:

Moseyenkov, B. I. Double rigidity rod oscillations in transitional torque. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 155–168. (Ukrainian. Russian and English summaries)

The transverse bending oscillations in transitional torque of a shaft with unequal principal stiffnesses are investigated. The differential equations of the problem are established taking into account the proper weight of the shaft, the force of friction and the static unbalance.

In the case when the two principal cross-section stiffnesses differ only slightly the equations are solved in the first approximation by the Krylov-Bogolyubov asymptotic method and then are applied to some special cases.

{Reviewer's remark: transitional torque refers to the transitional state of rotation, i.e., to the passage of the domain of parametric excitation with constant angular acceleration.} *T. P. Anđelić* (Belgrade)

2108:

Newton, Robert R. On the optimum trajectory of a rocket. J. Franklin Inst. **266** (1958), 155–187.

The author studies optimum rocket trajectories, based on equations of motion which incorporate the following assumptions: The entire trajectory lies in a fixed vertical plane; lift and drag have pseudo-static values, with lift and drag coefficients a function of Mach number; rocket orientation is always along the flight path (i.e., perfect weathercocking occurs); and the relative exhaust velocity of rocket jet is constant.

Variational techniques are used to study a variety of range problems, as well as problems relating to most efficiently placing the vehicle into an earth orbit.

The author's treatment is lucid and the mathematical results are closely integrated with their physical implications. *M. Goland* (San Antonio, Tex.)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 2046, 2047, 2196, 2197, 2198a-b, 2217, 2228, 2229.

2109:

Christov, Chr. Über die Wanderung der Moleküle in einem Gase. I. Acta Phys. Acad. Sci. Hungar. **6** (1956), 325–336. (Russian summary)

The random movement of a molecule as the result of collisions with other molecules in a gas is considered. The molecules are considered as hard elastic spheres of diameter a; and their distribution is assumed to be Maxwellian. From the dynamics of collisions between hard elastic spheres, it follows that the probability $P(\mathbf{u}, \mathbf{v})dtd\mathbf{v}$ that a molecule with a velocity $\mathbf{u}$ will experience exactly one collision during a time dt and will have its velocity changed to lie in the interval $(\mathbf{v}, \mathbf{v}+d\mathbf{v})$ is given by

$$(1)\qquad P(\mathbf{u}, \mathbf{v})dtd\mathbf{v}=\frac{a^2N}{c\pi^{\frac{1}{2}}}\exp[-(\mathbf{v}\cdot\mathbf{w})^2/c^2w^2]\,dtd\mathbf{v},$$

where $\mathbf{w}=\mathbf{u}-\mathbf{v}$, c is the root mean square velocity and N is the number of molecules per cm^3. The probability of the same event, without regard to the velocity subsequent to the collision, is given by

$$(2)\qquad pdt=a^2cNF(u/c)dt,$$

where

$$(3)\qquad F(s)=\pi^{\frac{1}{2}}\left\{e^{-s^2}+(2s+s^{-1})\int_0^s e^{-z^2}dz\right\}$$

(the foregoing formulae are familiar from Maxwell's discussion of the "persistence of velocities"). The probability, then, that a molecule with a velocity $\mathbf{v}_0$ at t_0 will experience during a time t exactly k collisions occurring during (t_1, t_1+dt_1), (t_2, t_2+dt_2), $\cdots$, (t_k, t_k+dt_k) with velocities respectively becoming $(\mathbf{v}_1, \mathbf{v}_1+d\mathbf{v}_1)$, $(\mathbf{v}_2, \mathbf{v}_2+d\mathbf{v}_2)$, $\cdots$, $(\mathbf{v}_k, \mathbf{v}_k+d\mathbf{v}_k)$ (the velocity after the last collision not being specified) is

$$(4)\qquad \prod_{j=0}^{k-1}[e^{-p(v_j)t_j}P(\mathbf{v}_j, \mathbf{v}_{j+1})dt_jd\mathbf{v}_j]e^{-p(v_k)t_k}dt_kd\mathbf{v}_k.$$

During such a sequence of collisions, the particle is displaced by the amount $\mathbf{r}=\sum_{j=1}^{k}\mathbf{v}_jt_j$. The probability $W_k(t, \mathbf{r}, \mathbf{v}_0, \mathbf{v})$, that after the result of k unspecified collisions during a time t the molecule with an initial velocity $\mathbf{v}_0$ will suffer a given displacement $\mathbf{r}$, can be found by integrating (4) over all $d\mathbf{v}_j$, dt_j after the insertion of the δ-function-factor

$$(5)\qquad \delta\Big(\sum_{j=0}^{k}\mathbf{v}_jt_j-\mathbf{r}\Big)\delta\Big(\sum_{j=0}^{k}t_j-t\Big)\delta(\mathbf{v}_k-\mathbf{v})$$
$$=\delta(\mathbf{v}_k-\mathbf{v})(2\pi)^{-4}\int_{-\infty}^{+\infty}d\omega\exp\Big[-i\omega\Big(\sum_{j=0}^{k}t_j-t\Big)\Big]$$
$$\times\int_{-\infty}^{+\infty}d\boldsymbol{\varkappa}\exp\Big[-i\boldsymbol{\varkappa}\cdot\Big(\sum_{j=0}^{k}\mathbf{v}_jt_j-\mathbf{r}\Big)\Big].$$

The integrations over the t_j's can be performed and we are left with the following expression for W_k:

$$(6)\qquad W_k(t, \mathbf{r}, \mathbf{v}_0, \mathbf{v})=(2\pi)^{-4}\iint F_k(\omega, \boldsymbol{\varkappa}, \mathbf{v}_0, \mathbf{v})e^{i\omega t+i\boldsymbol{\varkappa}\cdot\mathbf{r}}d\omega d\boldsymbol{\varkappa}$$

where

$$(7)\qquad F_k(\omega, \boldsymbol{\varkappa}, \mathbf{v}_0, \mathbf{v})=(p(v_0)+i\omega+i\boldsymbol{\varkappa}\cdot\mathbf{v}_0)^{-1}$$
$$\times\iint\cdots\int\delta(\mathbf{v}_k-\mathbf{v})\prod_{j=1}^{k}\frac{P(\mathbf{v}_{j-1}, \mathbf{v}_j)d\mathbf{v}_j}{p(v_j)+i\omega+i\boldsymbol{\varkappa}\cdot\mathbf{v}_j}.$$

The total probability that a particle will suffer a displacement $\mathbf{r}$ during a time t with unspecified numbers of collisions is given by

$$(8)\qquad W(t, \mathbf{r}, \mathbf{v}_0, \mathbf{v})=\sum_{k=0}^{\infty}W_k(t, \mathbf{r}, \mathbf{v}_0, \mathbf{v}).$$

Accordingly,

$$(9)\qquad W(t, \mathbf{r}, \mathbf{v}_0, \mathbf{v})=(2\pi)^{-4}\iint F(\omega, \boldsymbol{\varkappa}, \mathbf{v}_0, \mathbf{v})e^{i\omega t+i\boldsymbol{\varkappa}\cdot\mathbf{r}}d\omega d\boldsymbol{\varkappa},$$

where

$$(10)\quad F(\omega, \mathbf{x}, \mathbf{v}_0, \mathbf{v})=(p(v_0)+i\omega+i\mathbf{x}\cdot\mathbf{v}_0)^{-1} \times \sum_{k=0}^{\infty}\int_{\mathbf{v}_1}\int_{\mathbf{v}_2}\cdots\int_{\mathbf{v}_k}\delta(\mathbf{v}_k-\mathbf{v})\prod_{j=1}^{k}\frac{P(\mathbf{v}_{j-1},\mathbf{v}_j)d\mathbf{v}_j}{p(v_j)+i\omega+i\mathbf{x}\cdot\mathbf{v}_j}.$$

From the form of this last equation, it is apparent that F is a solution of the nonhomogeneous linear integral equation

$$(11)\quad [p(u)+i\omega+i\mathbf{x}\cdot\mathbf{u}]F(\omega,\mathbf{x},\mathbf{u},\mathbf{v})=\delta(\mathbf{u}-\mathbf{v}) + \int P(\mathbf{u},\mathbf{w})F(\omega,\mathbf{x},\mathbf{w},\mathbf{v})\,d\mathbf{w}.$$

The Fourier transform of this equation leads to the following integral equation for W:

$$(12)\quad W(t,\mathbf{r},\mathbf{u},\mathbf{v})=e^{-p(u)t}\delta(\mathbf{r}-\mathbf{u}t)\delta(\mathbf{u}-\mathbf{v}) + \int_{\tau=0}^{t}P(\mathbf{u},\mathbf{w})e^{-p(u)\tau}W(t-\tau,\mathbf{r}-\mathbf{u}\tau,\mathbf{w},\mathbf{v})d\tau d\mathbf{w}.$$

The derivation of this equation is the principal result of the paper.

In writing (4), we have essentially supposed that the process is Markovian; accordingly, we may write:

$$(13)\quad W(t+\Delta t,\mathbf{r},\mathbf{u},\mathbf{v})=\iint W(\Delta t,\mathbf{s},\mathbf{u},\mathbf{u}-\mathbf{w}) \times W(t,\mathbf{r}-\mathbf{s},\mathbf{u}-\mathbf{w},\mathbf{v})ds d\mathbf{w},$$

$$(14)\quad W(t+\Delta t,\mathbf{r},\mathbf{u},\mathbf{v})=\iint W(t,\mathbf{r}-\mathbf{s},\mathbf{u},\mathbf{v}-\mathbf{w}) \times W(\Delta t,\mathbf{s},\mathbf{v}-\mathbf{w},\mathbf{v})ds d\mathbf{w}.$$

Expanding with respect to Δt, $\mathbf{r}-\mathbf{s}$ and $\mathbf{u}-\mathbf{w}$ in Taylor series and integrating term by term, we obtain from (13) the following analogue of the Fokker-Planck equation:

$$(15)\quad \sum_{k=0}^{\infty}\Delta t^k W_{t^k}/k!=\sum_{m,n=0}^{\infty}\frac{(-1)^m}{m!}\frac{(-1)^n}{n!}A_{mn}(\Delta t,\mathbf{u})W_{\mathbf{r}^m\mathbf{u}^n}$$

where

$$(16)\quad A_{mn}(\Delta t,\mathbf{u})=\iint \mathbf{s}^m\mathbf{w}^n W(\Delta t,\mathbf{s},\mathbf{u},\mathbf{u}-\mathbf{w})ds d\mathbf{w},$$

where the indices r^m, u^n and t^k indicate m times, n times and k times, differentiations with respect to r, u and t, respectively; also the quantities $\mathbf{s}^m\mathbf{w}^n$, A_{mn} and $W_{\mathbf{r}^m\mathbf{u}^n}$ are to be considered as tensors of rank $m+n$. By treating equation (14) similarly, we obtain the adjoint of (15).

Using the expression for W already derived, the author shows that A_{mn} is of the form $A_{mn}{}^0+A_{mn}{}^1$. For $\Delta t\to 0$, these basic tensors reduce to

$$A_{00}{}^0=1-p(u)\Delta t,\quad A_{10}{}^0=\mathbf{u}\Delta t,$$

$$A_{mn}{}^1=\begin{cases}(-1)^n\Delta t A_n(\mathbf{u}) & \text{for } m=0,\\ 0 & \text{for } m>0,\end{cases}$$

where

$$A_n(\mathbf{u})=\int \mathbf{w}^n P(\mathbf{u},\mathbf{u}+\mathbf{w})d\mathbf{w}.$$

All the other coefficients are of higher order in Δt. With these expressions (15) becomes

$$(17)\quad W_t+\mathbf{u}\cdot W_{\mathbf{r}}=\sum_{n=1}^{\infty}(n!)^{-1}A_n(\mathbf{u})\cdot W_{\mathbf{u}^n}.$$

The paper concludes with the explicit evaluation of A_n for $P(\mathbf{u},\mathbf{v})$ given by (1). {The reviewer has felt justified in making so extensive a review of this paper since it appears to him as of fundamental importance and the original may not be generally available.}

S. Chandrasekhar (Williams Bay, Wis.)

2110:

Christov, Chr. Über die Wanderung der Moleküle in einem Gase. II. Acta Phys. Acad. Sci. Hungar. 7 (1957), 51–66. (Russian summary)

In this paper the author used the developments of his earlier paper [see preceding review] to obtain expressions for the coefficients of dynamical friction and diffusion characterizing macroscopically the random movement of a molecule in a gas. Thus,

$$\left(\frac{d\mathbf{u}}{dt}\right)_{\mathrm{av}}=\lim_{t\to 0}t^{-1}\iint(\mathbf{v}-\mathbf{u})W(t,\mathbf{r},\mathbf{u},\mathbf{v})d\mathbf{r}d\mathbf{v} =\int \mathbf{w}P(\mathbf{u},\mathbf{u}+\mathbf{w})d\mathbf{w}=-a^2Nc^2F_1{}^0(u/c)\mathbf{u},$$

where

$$F_1{}^0(s)=\pi^{\frac{1}{2}}\left\{e^{-s^2}\left(\tfrac{1}{2}s+\frac{1}{4s}\right)+\left(s^2+1-\frac{1}{4s^2}\right)\int_0^s e^{-x^2}dx\right\}$$

(for the meaning of the symbols, see the preceding review). In the framework of the author's theory, it is not possible to define a diffusion coefficient in an unambiguous way. However, by writing the mean displacement $\mathbf{r}$ which a molecule with a velocity $\mathbf{u}$ experiences in time t as the average of the displacements suffered after one, two, etc., collisions, we can write

$$\overline{\mathbf{r}(\mathbf{u})}=\frac{\mathbf{u}}{p(u)}+\int\frac{P(\mathbf{u},\mathbf{v})}{p(u)}\frac{\mathbf{v}}{p(v)}d\mathbf{v} + \iint\frac{P(\mathbf{u},\mathbf{v})}{p(u)}\frac{P(\mathbf{v},\mathbf{w})}{p(v)}\frac{\mathbf{w}}{p(w)}d\mathbf{v}d\mathbf{w}+\cdots.$$

Therefore,

$$\overline{\mathbf{r}(\mathbf{u})}=\frac{\mathbf{u}}{p(u)}+\int\frac{P(\mathbf{u},\mathbf{v})}{p(u)}\overline{\mathbf{r}(\mathbf{v})}d\mathbf{v}.$$

By writing $[\mathbf{r}(\mathbf{u})^2]_{\mathrm{av}}=\alpha^2u^2$, we obtain

$$\alpha^2\int u^2V(u)du=\int\overline{\mathbf{r}(\mathbf{u})}^2V(u)d\mathbf{u},$$

where $V(u)$ is the assumed Maxwellian distribution of velocities. Using the foregoing equation, the author shows that we may define a quantity analogous to the usual diffusion coefficient; this is given by $D=\frac{1}{6}\alpha c^2=\gamma c/a^2N$, where $\gamma=0.151$. A somewhat more rigorous discussion leads to an expression of the same form but with $\gamma=0.135$.

The paper concludes by devising an iteration scheme (based on the foregoing expression for dynamical friction and diffusion) which will solve explicitly for the basic probability function $W(t,\mathbf{r},\mathbf{u},\mathbf{v})$.

S. Chandrasekhar (Williams Bay, Wis.)

2111:

Christov, Chr. Über die Wanderung der Moleküle in einem Gasgemisch. Acta Phys. Acad. Sci. Hungar. 7 (1957), 67–85. (Russian summary)

The present paper generalizes the analysis of two earlier papers [see preceding reviews] for the case when the molecule is randomly moving through a mixture of gases. The author considers the case when the different gases are composed of hard elastic spheres of differing diameters. The case of a binary mixture is considered in great detail. The analysis follows the general lines of the earlier papers; and as no new additional principles are involved, no attempt will be made to describe the complicated formal analysis.

S. Chandrasekhar (Williams Bay, Wis.)

2112:
Das Gupta, Santi Ranjan. On the solution of transport equation for neutron diffusion in the Milne-Eddington model. Indian J. Theoret. Phys. **5** (1957), 39–48.

The diffusion of neutrons in a semi-infinite plane parallel medium with a law of scattering which includes the first two spherical harmonics is considered. The relevant equation of transfer is solved by the method of Laplace transforms and the angular distribution of the emergent radiation is expressed in terms of H-functions [cf. S. Chandrasekhar, Radiative transfer, Clarendon Press, Oxford, 1950; MR **13**, 136].

S. Chandrasekhar (Williams Bay, Wis.)

2113:
Newell, G. F. Some statistical problems encountered in a theory of pinning and break away of dislocations. Quart. Appl. Math. **16** (1958), 155–168.

The author considers two kinds of statistical problems having their origin in the motions of pinned dislocation lines in crystals. He distributes points (the impurity atoms) at random along a line (the dislocation line), the distance between consecutive points following the Poisson distribution law. He then extracts from this line segments of length Z (the endpoints of which, the so-called strong pins of the dislocation line, do not participate in the motion of the line) and assumes Z to have a preassigned probability density $P(Z)$. He then determines: (1) The probability that in this segment no pair of second nearest neighbors are separated by a distance greater than some preassigned distance D; (2) the probability distribution for the distance between consecutive points for those segments satisfying the condition in (1).

Solutions are expressed in the form of single contour integrals involving the Laplace transform of $P(Z)$. In the particular case that $P(Z)$ is expressible in the form $P(Z)=\sum_{i=1}^{n} p_i(z)\rho^{-a_i Z}$, where each $p_i(Z)$ is a polynomial, the final integrations can be carried out explicitly, yielding exact solutions in closed form.

H. A. Hauptman (Washington, D.C.)

2114:
Brinkman, H. C. Non-linear irreversible thermodynamics of Brownian motion. Physica **24** (1958), 409–414.

Irreversible thermodynamics is studied for a special system far from equilibrium. Kramers' theory of Brownian motion leads to a diffusion equation in phase space. The distribution function resulting from this equation is used for the definition of free energy as a function of time. A system of non-linear irreversible thermodynamics is developed on this basis. Relations which are analogous to the Onsager relations are found. The formalism appears to be applicable to a much wider class of diffusion equations. For these more general cases also, Onsager's relations appear automatically, while microscopic irreversibility has not been introduced explicitly. (Author's abstract.)

D. Falkoff (Waltham, Mass.)

2115:
van Vliet, K. M. Irreversible thermodynamics and carrier density fluctuations in semiconductors. Phys. Rev. (2) **110** (1958), 50–61.

The coefficient in the statistical theory of irreversible processes due to Onsager and Machlup [Phys. Rev. **91** (1953), 1505–1512; MR **15**, 273] are evaluated for the case of carrier density fluctuations in semiconductors using a kinetic model. *D. Falkoff* (Waltham, Mass.)

2116:
Callen, Herbert B. Path distribution for irreversible processes. Phys. Rev. (2) **111** (1958), 367–372.

An expression is derived for the conditional probability for the change in time of a fluctuating thermodynamic variable when an external time-dependent force is applied. This generalizes the statistical theory of irreversible processes of L. Onsager and S. Machlup [same Rev. **91** (1953), 1505–1512; MR **15**, 273], N. Hashitsume [Progr. Theoret. Phys. **8** (1952), 461–478; MR **14**, 1048] and H. B. Callen and R. F. Greene [Phys. Rev. (2) **86** (1952) 702–710; MR **14**, 230] by giving the statistical properties of non-equilibrium macroscopic states.

D. Falkoff (Waltham, Mass.)

ELASTICITY, PLASTICITY

See also 2136, 2137.

2117:
***Kröner, Ekkehart. Kontinuumstheorie der Versetzungen und Eigenspannungen.** Ergebnisse der angewandten Mathematik. Bd. 5. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. vii+179 pp.

In 1934 Taylor explained the plastic deformation of metals on the basis of dislocation theory. A lot of work followed in both the macroscopic and microscopic fields. The present book treats dislocation theory from the continuum view point.

The continuum theory of dislocations has played an important role in bridging the gap between the phenomenological and atomic theories of plasticity. The author rightly says that this theory can be considered a significant contribution to scientific research in this century. The book follows the lines of R. Grammel's collection "Verformung und Fliessen des Festkörpers" [Springer, Berlin-Göttingen-Heidelberg, 1956; MR **19**, 336], which treats both the mechanical and mathematical aspects, on the one hand, and the solid state physics aspect on the other.

The physical ideas of edge and screw dislocations, slip vector and glide planes, Burger's vector and the analogy between thermal (included here under the type called quasi-plastic) deformation and plastic deformation are explained, with figures, in the introduction.

The relations between the geometry of deformation and the dislocations is treated in the first chapter. After explaining the connection between Volterra's distortion and dislocations and proving that the lines of dislocation must either be closed loops or must end at the surfaces only, the author introduces the total (elastic and plastic) distortion tensor (β_{ij}^T) as $ds_j^T=\beta_{ij}^T dx_i$. Using the condition that connectivity of the body is preserved, it is shown that Rot $\beta^T=0$. Defining dislocation density as $\bar{\alpha}=(\alpha_{ij})=-\text{Rot}\,\beta^p$ (where β^p is plastic component of β^T), the basic geometric equation of continuum mechanics $\bar{\alpha}=\text{Rot}\,\bar{\beta}$ is deduced ($\bar{\beta}$ being the elastic component). Since it is the moving dislocations that cause slip and hence plastic flow, a tensor N_{ijk} is introduced to represent the movement of α_{jk} dislocations in the i-direction, $j=k$ and $j\neq k$ giving screw and edge dislocations.

Then follows the decomposition of distortion into components as gradient and curl of two parts and further into the form

$$\nabla_i S_j' - \varepsilon_{ikl}\varepsilon_{jmn}\nabla_k\nabla_m i_{ln} + \theta_{ij} = \text{Grad}\,\bar{s} + \text{Ink}\,i + \bar{\theta},$$

the compatibility conditions of classical theory being given by $\mathrm{Ink}\,\varepsilon=0$. The incompatible deformation fields arising in thermal and magnetic fields are then discussed. Finally we get discussions about structural rotations, large deformations and the determination of the distortion of a substance with dislocations.

The second chapter deals with the statical viewpoint. Volume and surface dislocations are treated. By use of the stress-function approach the energy of volume distributions is obtained and the particular case of two volume distributions is noted, giving self and mutual energies. Expanding the method of obtaining general solutions of classical elasticity theory (with anisotropy) the displacement fields for line and surface distributions are obtained. It is shown that a field arising from dislocations can be considered, similar to the magnetic field, as due to either line distributions or surface distributions of dislocations. The case of singular dislocations is then discussed and from classical elasticity theory self and mutual energies are obtained, noting their analogy to self and mutual inductances. Finally, stress fields due to dislocations are treated. By use of the principle of virtual displacement the formula of Peach and Koehler $d\bar{K}=d\bar{L}x^{\delta}\cdot\bar{b}$ is obtained, $d\bar{K}$ being force, $d\bar{L}$ displacement, x^{δ} stress and $\bar{b}$ Burger's vector. This is applied to obtain stress and displacement fields due to different types of multiple force and displacement singularities. Analogies with the electro-magnetic field are frequently noted.

We then get the discussion on dislocations in crystals. The basic concepts in the order of magnitude of infinitesimals involved in the mathematical treatment of the microscopic region and of the passage to the macroscopic are discussed. Starting with Frank's definition of dislocation in crystals, displacements, distortion and deformation in the microscopic are explained leading to $\delta_{\mathrm{Rot}\,\bar{\beta}=\bar{\alpha}}$ as before, showing the existence of elastic distortion in presence of dislocations. With this the author passes to the macroscopic region and establishes relevant equations. Then the relations between the grain-boundary orientations and dislocation arrangements are discussed. The resulting relations giving the type and density of dislocations which give stresses or no stresses are deduced. Dislocation types in cubic face-centred crystals are then discussed. Finally Peierl's and Eshelby's treatments of edge and screw dislocations for primitive cubic crystals, and Liebfried and Dietze's treatment for face centred cubic crystals is given in detail.

The fourth chapter deals with the non-Riemannian geometry of dislocations. Kundo's and Bilby-Bullough-Smith's (B.B.S.) theories are first discussed.

Defining the Cartan's torsion tensor $(\Gamma^{\kappa}{}_{(\lambda\mu)})$, Riemann-Christoffel curvature tensor $(R^{\kappa}{}_{\lambda\mu\nu})$ and Euler-Schouten's curvature tensor $(H_{ij}{}^{\Lambda})$, it is shown that the dislocation density can be related to torsion as $\alpha^{\lambda\kappa}=\varepsilon^{\lambda\mu\nu}\Gamma^{\kappa}{}_{[\mu\nu]}$. Then Kundo's classification of lattice flaws as (i) those with incompatible metric with non-vanishing curvature in natural state (points of curvature flaw) (ii) non-Riemannian lattice defects with non-vanishing torsion in natural state (points of torsion flaws) (iii) Lattice flaws with non-vanishing Euler-Schouten's curvature, and the agreement with the ideas in the book is then noted. The B.B.S. theory is then given. Starting with the geometry of deformation of lattices relations between the geometric quantities and local and true Burger's vector (identical only for small deformation), torsion tensor and dislocation density and curvature tensor and stress-free state are discussed. The relation between Nye's curvature tensor and torsion in non-Riemannian geometry is then explained. The chapter closes with a discussion of how Kundo's and B.B.S.'s theories pertain to the macroscopic and microscopic regions, respectively, how a new theory of plasticity can be built up from non-Riemannian geometry and the relations between the virtual (Kundo's), real and local densities of dislocations.

In the last chapter we have some applications.

The first one is the interesting but difficult problem of work-hardening of metals. Here we get an outline of the case of cubic face centred metals. The stress-strain curve is divided into 3 parts.

The small amount of hardening in the first region is easily explained. For the second region where hardening coefficient is large, it is explained on the basis of the occurrence of Lomer-Cotterell dislocations when a Frank-Read source of dislocation meets another on a different slip-plane. The case of small coefficient in the last region, it is noted, is still not solved satisfactorily. The explanation of 'active energy' is given, and a method to calculate it.

The next application is to the approximate calculation of self-energy of curvilinear dislocations.

Then the notion of interstitial or foreign atoms acting as dipoles and polarisation centres is explained. Interactions of these and other lattice faults with dislocations are treated mathematically.

Finally, as an application of stress function tensors to three dimensional rotationally symmetric problems, the solution of a new problem of interaction of two circular dislocations is given.

The book ends with a bibliography which lists 180 items. Physicists, mathematicians and engineers will all find the book interesting. *B. R. Seth* (Kharagpur)

2118:

Stoppelli, Francesco. Sull'esistenza di soluzioni delle equazioni dell'elastostatica isoterma nel caso di sollecitazioni dotate di assi di equilibrio. Ricerche Mat. **6** (1957), 241–287.

The author continues his effective program of establishing the existence, uniqueness, and analyticity of solutions to the stress boundary-value problem of finite hyperelasticity. In earlier papers he has disposed of the general case, when the applied loads do not have an axis of equilibrium [same Ricerche **3** (1954), 247–267; **4** (1955), 58–73; MR **17**, 554, 801]. The peculiar difficulty of the case when there is an axis was revealed in researches on formal expansions by Signorini [Atti 24 Riun. Soc. Ital. Progr. Sci. **3** (1936), 6–25, and later papers]. Here, one or more of the problems of linear elasticity to which the steps of the perturbation series correspond may fail to be compatible. Tolotti attempted to determine conditions that some one or more orientations of the body be such that a solution exists; an orientation such that the first $n+1$ linear problems are compatible he called a "principal orientation of nth order" [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (7) **13** (1943), 1139–1162; MR **7**, 141].

In the present elaborate work the author begins by replacing the above purely formal conclusions by rigorous analytic theorems. He defines a displacement $\mathbf{s}$ as being "vanishing in the strong sense at $\theta=0$" if $\mathbf{s}$ along with its first and second derivatives and Hölder coefficients of these latter are infinitesimal as $\theta\to 0$, θ being the multiplicative factor for the assigned loads as in the previous works on this subject.

The author proves two major theorems. (A) A necessary

condition for a solution vanishing in the strong sense is that the body be in a principal orientation of first order. (B) When all orientations under which the load remains equilibrated are principal orientations of first order, a necessary condition for a solution vanishing in the strong sense is that the body be in a principal orientation of second order. The interest in these theorems derives principally from the fact that the solutions in these cases are not to be expected to turn out to be analytic functions of θ.

The analysis, difficult in itself, is obscured by the notations used by the Italian school. Further memoirs are to follow. *C. Truesdell* (Bloomington, Ind.)

2119:

Yoffe, Elizabeth H. The centre of a dislocation. II. The dilated slit. Phil. Mag. (8) **3** (1958), 8–18.

[For part I, see same Mag. **2** (1957), 1197–1210].

An isotropic elastic body in a state of plane stress is considered, employing stress functions that give discontinuities in the y component of the displacement across the cut $-h \leq x \leq h$; the dislocations can thus be supposed caused by the insertion of additional material of given thickness $d(x)$ in a line crack. The graphs of the functions $d(x)$ are made up of arcs of parabolas and straight-lines; in the two special cases considered in detail, the stress systems, strain energies and density changes are calculated. *W. R. Dean* (London)

2120:

Smith, G. F.; and Rivlin, R. S. The strain-energy function for anisotropic elastic materials. Trans. Amer. Math. Soc. **88** (1958), 175–193.

Allowing the strain energy function to be an arbitrary polynomial in the components of finite strain, the authors determine explicitly the restrictions imposed when the elastic material belongs to one of the 32 crystal classes. 11 different types of strain energy function emerge; in the classical linear approximation, these coalesce into 9. The basic algebraic problem is determination of a polynomial basis for the set of polynomials that are form-invariant under the group of transformations which define the crystal class in consideration. The proofs follow from suitable application of certain results in the classical theory of groups. *C. Truesdell* (Bloomington, Ind.)

2121:

Hacar Benitez, Miguel-Angel. Application of finite difference equations to the approximate solution of problems in elasticity and strength of materials. Las Ciencias **22** (1957), 563–603. (Spanish)

Il titolo stesso della Nota esprime un procedimento di calcolo non del tutto nuovo, quantunque oggidì poco usato, per il quale all'uso di funzioni derivate si sostituisce quello del rapporto fra la differenza di valori prossimi di data funzione e quella dei corrispondenti valori della variabile direttiva; e ben si sa che in casi di pratica applicazione tal sostituzione offre con maggior semplicità risultati del tutto soddisfacenti. — Tal punto di vista risulta precisamente indicato nelle prime 5 pagine della Nota presente, alle quali seguono alcuni schiarimenti con rispetto alla effettiva sostituzione anzi indicata di eventuali derivate prime e seconde mediante semplici rapporti numerici, dope di che vengono segnalati 6 particolari esempi ordinatamente relativi al caso di una sbarra con carica uniforme appoggiata agli estremi, quindi al caso in cui gli estremi della sbarra si suppongono incastrati, per passare quindi al caso di una piastra quadrata ancora uniformemente caricata e appoggiata ai lati e al caso analogo di piastra triangolare, terminando finalmente con due problemi di maggior analisi, relativi rispettivamente al ipotetico torcimento di un'asta e all'ipotetica inflessione di una sbarra o colonna verticale. *B. Levi* (Rosario)

2122:

Zanaboni, Osvaldo. Soluzione mediante serie delle travi snelle pressoinflesse. Ann. Mat. Pura Appl. (4) **44** (1957), 293–315.

The problem of a continuous beam, with any number of sections, and each section with bending stiffness EJ either constant or linearly variable, and under an axial stress S and a transversal load either uniformly distributed or concentrated in one or more points, has been discussed by L. Cesari, F. Conforto, and C. Minelli [Travi continue inflesse e sollecitate assialmente, Istituto per le applicazioni del calcolo, Roma, 1941; MR **16**, 769] (the coefficients of the final three-moment equations can be read off without effort from numerical tables whose lengthy preparation involved Bessel functions for EJ linearly variable). In the present paper the author rediscusses the whole problem making use of the known fact that for each section the eigenfunctions of the equations $(EJx'')''+Sx''=0$ with corresponding boundary conditions are orthogonal and allow series developments analogous to Fourier series. The two examples given concern cases with EJ constant. {The difficulties inherent in the determination of eigenvalues and eigenfunctions when EJ is not constant are not mentioned.} *L. Cesari* (Baltimore, Md.)

2123:

Zorski, Henryk. Plates with discontinuous supports. Arch. Mech. Stos. **10** (1958), 271–313. (Polish and Russian summaries)

Author considers bending problems for a semi-infinite elastic plate in which part of the straight boundary is clamped and part is simply supported. Solution is effected by integral equations. The same method of solution is then used for a plate in the region $x>0$, $y>0$ one edge of which is simply supported and the other partially clamped and partially simply supported. *A. E. Green* (Newcastle-upon-Tyne)

2124:

Zorski, Henryk. A semi-infinite strip with discontinuous boundary conditions. Arch. Mech. Stos. **10** (1958), 371–398. (Polish and Russian summaries)

This paper is a continuation of the one reviewed above. Using similar methods, the author considers the bending of a semi-infinite strip which is simply supported on the long edges $x=\pm 1$ and partially clamped and partially simply supported on the short edge $y=0$. *A. E. Green* (Newcastle-upon-Tyne)

2125:

Dnestrovskiĭ, Yu. N. Variation of normal frequencies of membranes and resonators with additional loads. Akust. Ž. **4** (1958), 244–252. (Russian)

The question of how the normal frequencies of a loaded vibrating membrane or cavity resonator change when either the load is increased without changing the region of application or the region is contracted without changing the load is treated. It is found that the essential parameter for determining the answer is the "capacity" of the region.

Specifically, let the membrane or resonator occupy a domain T with boundary Γ, and let λ_k, u_k be the respective k-th eigenvalue and eigenfunction of the problem:

$\Delta u+\lambda\rho u=0$ in T, $u=0$ on Γ. Let the density ρ vary inside some region gCT with boundary γ, and let it be denoted by $\bar{\rho}$ in this region. The integrals $m=\int_g(\bar{\rho}-\rho)d\tau$ and $C(g,\Gamma)=(4\pi)^{-1}\int_\gamma(\partial S^0/\partial n)d\sigma$ are the load and capacity, respectively, S^0 being harmonic in $T-g$ with the values 0 on Γ and 1 on γ. It is required to find the eigenvalues $\lambda^{(k)}$ and eigenfunctions $u^{(k)}$ of the "fundamental" problem: $\Delta u+\lambda\bar{\rho}u=0$ in T, $u=0$ on Γ. If T is such that this problem has a Green's function, then the resulting integral equation for u can be solved by successive approximations.

The principal results are the following: (i) if the load is concentrated, i.e., if it is applied to a region of capacity zero, then the solution to the fundamental problem does not exist; (ii) if a constant load is applied to a region which contracts to a set of zero capacity, then the entire spectrum shifts to the left in the sense that $\lim_{C\to 0}\lambda^{(k)}\leq\lambda_{k-1}$; (iii) if a large load is applied to a region of small capacity, each eigenvalue again shifts to the left, but by an amount which depends on the particular eigenvalue and the capacity; (iv) if a small load is applied to a region of small capacity, the spectrum is perturbed in the same direction, and an estimate for the smallest eigenvalue is obtained which refines the results of Rayleigh's theory [Theory of sound, Macmillan, London, 1937; v. 1, p. 335; for a review of 2nd edition, see MR 7, 500].

R. N. Goss (San Diego, Calif.)

2126:

Cicala, Placido. Sulla teoria della lastra elastica ortotropa avente superficie media di rivoluzione o cilindrica. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **92** (1957–58), 30–64.

Giá per un breve riassunto anteposto a questa Nota appare trattarsi in essa di offrire particolari informazioni e caratteristiche indicazioni circa la "superficie media" di uno strato curvo con determinato asse di rotazione; ed effettivamente, fin dalle prime linee dell'Introduzione, allude l'A. alla ormai classica "Mathematical theory of elasticity" di A. E. H. Love [4th ed., Dover Publ., New York, 1944; MR 6, 79]. Nondimeno una differenza essenziale è da notare in confronto a questo testo ed è la qui relativa assenza di calcoli effettivi, generalmente sostituiti mediante formula riassuntive. — Consta precisamente la Nota di 3 parti (o capitoli) dei quali il primo intende determinare le equazioni per il movimento di un punto generico e corrispondenti forze virtuali del corpo in considerazione, mentre la parte seconda si riferisce alle condizioni limiti della lastra di spessore evenescente, e la terza parte sostanzialmente appare riassumere in forma preferentemente descrittiva particulari risultati con relative allusioni a precendenti ricerche di noti autori, come Geckeler, Reissner, Havers, Jakobsen. Termina nondimeno questa serie di osservazioni affermando che: "I risultati ottenuti suggeriscono l'opportunità di estendere l'applicazione di questi concetti ad altre classi di problemi attinenti alle pareti sottili." *B. Levi* (Rosario)

2127:

Bašeleišvili, M. O. On fundamental solutions of the differential equations of an anisotropic elastic body. Soobšč. Akad. Nauk Gruzin. SSR. **19** (1957), no. 4, 393–400. (Russian)

Lamé field equations of equilibrium and vibration of orthotropic media are given and components of displacement, for some restricted types of orthotropy, are represented in terms of one strain function satisfying a sixth order differential equation. Fundamental solutions of this equation are found, the fourth derivatives of which possess the character of the function $1/r$. As a particular case of the author's solution, Kroener's fundamental solution for equilibrium of transversely isotropic medium is derived. *J. Nowinski* (Madison, Wis.)

2128:

***Mindlin, R. D.; and Deresiewicz, H. Timoshenko's shear coefficient for flexural vibrations of beams.** Proc. Second U. S. Nat. Congress of Applied Mechanics. University of Michigan, Ann Arbor, Michigan, June 14–18, 1954. pp. 175–178.

The shear coefficient in Timoshenko's equations for flexural vibrations of beams depends on the shape of the cross-section and on the mode of motion. The authors take the latter into account in deriving the shear coefficient for infinite beams of various cross-sections by equating solutions from Timoshenko's equations and the three-dimensional equations of elasticity for simple thickness-shear motions. (From author's summary.)

G. B. Warburton (Edinburgh)

2129:

Sen, Bibhutibhusan. Note on some problems of thin equilateral triangular plate. Indian J. Theoret. Phys. **5** (1957), 77–79.

The author discusses two vibrations problems involving simply supported equilateral triangular plates. In the first case the plate is subject to uniform tension, and in the second it rests on an elastic foundation. The solution is carried out in terms of trilinear coordinates, that is, the perpendicular distances of a generic point from the three sides of the triangle are used as the coordinate system. The author assumes a displacement function which satisfies the boundary conditions and, by forcing it to satisfy the differential equation as well, obtains the frequency equation. *W. Boyce* (Troy, N.Y.)

2130:

Green, A. E.; and Rivlin, R. S. The mechanics of nonlinear materials with memory. Arch. Rational Mech. Anal. **1** (1957), 1–21.

The authors examine materials subject to a continuously varying deformation in the time interval $(0, t)$ in which the stress components at the current time t are functions of the displacement gradients at time t and at N previous instants during the interval $(0, t)$. Limitations are imposed upon the form of the stress deformation relations by the consideration that the stress distribution must be unaffected by rigid body rotations of the entire system. Further restrictions necessary for isotropic materials are derived from the condition for form invariance under rotations of the initial reference frame.

If the number N is allowed to tend to infinity, so that the stress depends upon all states of deformation occurring in the interval $(0, t)$, certain functions in the stress deformation relations are replaced by functionals which, to any desired degree of approximation, may be replaced by the sum of a finite number of multiple integrals. Sufficient conditions on these integrals are then derived for the stress deformation relations to be of the hereditary type, the restriction imposed being that the stress shall depend only upon previous strain history and not upon the instant chosen for the commencement of the deformation. The connexion is examined between the present theory and that discussed by R. S. Rivlin and J. L. Ericksen [J. Rational Mech. Anal **4** (1955), 323–425; MR **16**, 881] in which the stress is assumed to be a function of

the displacement, velocity and first n acceleration gradients at current time t. The theory is initially formulated in rectangular Cartesian coordinates, but the authors indicate how their relations may be transformed to a general curvilinear system convected with the material.

J. E. Adkins (Nottingham)

2131:

Storchi, Edoardo. Linearizzazione del problema plastico ristretto della deformazione piana. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **91** (1957), 778–788.

Two parameters $\omega(x, y)=(\sigma_{xx}+\sigma_{yy})/4k$ and $\theta(x, y)=\frac{1}{2}\arctan((\sigma_{yy}-\sigma_{xx})/2\sigma_{xy})$ describe the stress field in a plastic rigid non-hardening material in plane strain. The author shows that, provided the Jacobian $\partial(\omega, \theta)/\partial(x, y)\neq 0$, $u=y\sin\theta+x\cos\theta$, $v=y\sin\theta-x\cos\theta$ both satisfy $\partial^2 f/\partial\omega^2-\partial^2 f/\partial\theta^2=f$. No examples are given; reviewer is unable to judge whether the result is of any practical value.

D. R. Bland (Manchester)

2132:

Chen, Shih-Yuan. Comments on "Investigation of the temperature distribution and thermal stresses in a hypersonic wing structure". J. Aero. Sci. **24** (1957), 544–545.

This note points out several errors which appeared in the analysis of the work by M. A. Goldberg [J. Aero. Sci. **23** (1956), 981–990; MR **18**, 776]. The proper correction is here provided.

T. Yao-tsu Wu (Pasadena, Calif.)

2133:

Thomas, T. Y. On the velocity of formation of Lüders bands. J. Math. Mech. **7** (1958), 141–148.

A Lüders band, such as is found in a flat tension specimen of a metal with a stress drop at yielding, is idealized as a plastic region in a non-hardening elastic-plastic solid (Prandtl-Reuss). The author obtains the result that the surface of the band is perpendicular to the flat sides of the specimen, and either perpendicular or parallel to the applied load, and that its normal velocity of propagation is equal to that of an equivoluminal wave in the elastic region.

R. Hill (Nottingham)

STRUCTURE OF MATTER

See also 2117, 2196, 2197, 2198a-b.

2134:

†Raaz, Franz; und Tertsch, Hermann. Einführung in die geometrische und physikalische Kristallographie. 3te Aufl. Springer-Verlag, Wien, 1958. xii+367 pp. $11.45.

Third edition of a book first published in 1939 by Springer. The first part, by Raaz, contains 202 pages and 19 chapters, with three main sections: Morphological crystallography; Crystallography of the discontinuum (=lattice and space-group theory); Structure determination and X-ray crystallography. The second part, by Tertsch, contains 154 pages and 15 chapters, and is also divided into three sections: Phenomena of solidity (elasticity, plasticity, cleavage, hardness); Crystal optics; Other physical properties (thermal, electrical and magnetic properties, density). {The treatment of the subjects of which the reviewer has special knowledge dates from 1939 rather than 1957.}

A. J. C. Wilson (Cardiff)

2135:

Emersleben, Otto. Über lineare Beziehungen zwischen Madelungkonstanten. Bülgar. Akad. Nauk. Izv. Mat. Inst. **2** (1957), no. 2, 87–120. (Bulgarian and Russian summaries)

Unter Anwendung einer neueren Beziehung zwischen Epsteinschen Zetafunktionen, insbesondere dem Bornschen Grundpotentials $\pi(x, y, z)$, auf ältere Ergebnisse des Verfassers erkennt man, dass die Madelungkonstanten der 4 Gittertypen NaCl, CsCl, ZnS und CaF_2 nur von 2 Zetafunktionswerten abhängen, von denen ausser $\pi(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ der Madelungkonstanten des Steinsalzgitters, jetzt auch der Wert von $\pi(0, 0, \frac{1}{2})$ mit einer den vergrösserten Ansprüchen entsprechenden Genauigkeit (12 Dezimalstellen) vorliegt. — Zwischen den Madelungkonstanten jener 4 Gittertypen bestehen lineare Beziehungen mit ganzzahligen Koeffizienten — z.B. zwischen je 3 dieser 4 Grössen je 1 lineare Beziehung. Von diesen Beziehungen sind 2 von einander linear abhängig. Daher kann man die Madelungkonstanten dieser 4 Gitter rational auf 2 unter ihnen zurückführen, z.B. auf diejenigen des Steinsalz- und des CsCl-Gitters. Letztere, die bisher nur als Summe zweier Zetafunktionswerte dargestellt war, wird jetzt auch durch einen einzelnen Zetafunktionswert ausgedrückt, mit einer rhomboedrischen quadratischen Form anstelle der bisherigen kubischen Formen.

W. Nowacki (Bern)

2136:

Bilby, B. A.; Bullough, R.; Gardner, L. R. T.; and Smith, E. Continuous distributions of dislocations. IV. Single glide and plane strain. Proc. Roy. Soc. London. Ser. A. **244** (1958), 538–557.

Die Theorie der kontinuierlich verteilten Versetzungen wurde von Bilby und seine Mitarbeiter schon in mehreren in dieser Zeitschrift besprochenen Artikeln ausgearbeitet. [Bilby, Bullough u. Smith, dieselben Proc. **231** (1955), 263–273; MR **17**, 687; Bilby u. Smith, ebenda **236** (1956), 481–505; MR **18**, 430; auch folgendes Referat.] In der zuletzt zitierten Arbeit besprechen die Verfasser das Problem der Verdrehung eines einachsigen Kristalles, in der vorliegenden — noch vor der erwähnten erschienen Arbeit — dagegen die Frage der ebenen Verbiegung von Kristallen.

Zuerst wird der Begriff der einzelnen Gleitung eingeführt, worunter die Verfasser Gleitungen entlang von ursprünglich zueinander parallelen Netzebenen des Kristalles verstehen. Im weiteren Teil der Arbeit wird nur dieser Fall ausführlich besprochen. In jedem Punkte P des deformierten Kristalls wird eine lokale Basis $\mathbf{e}_\alpha$ eingeführt und ausserdem ein Cartesisches Koordinatensystem $\mathbf{x}_\alpha$ für den ganzen Kristall. Dann folgt

$$\mathbf{e}_\alpha(P)=D_\alpha{}^\alpha(\mathbf{e}P\mathbf{x})\mathbf{x}_\alpha, \tag{1}$$

wo die Matrix $D(\mathbf{e}P\mathbf{x})$ die Deformation beschreibt. Die das lokale System beschreibenden Koordinaten sind im allgemeinen Fall nicht integrierbar (nichtholonom). Weiter wird angenommen, dass keine reine Gitterspannung auftritt, die D Matrizen beschreiben dann Rotationen. Danach wird der Burgerssche Vektor für die Flächeneinheit und der Tensor der Gitterkrümmung eingeführt. Nach diesen Vorbereitungen wird der spezielle Fall der einzelnen Gleitung in der $(\mathbf{x}_1, \mathbf{x}_3)$-Ebene betrachtet und es wird bewiesen, dass die Gleitflächen in diesem Falle abwickelbar sind, also verschwindet ihr Gauss-sches Krümmungsmass. Die Gleitflächen erhält man dann aus den Gleichungen $e_2(x_1, x_2, x_3)=\text{const}$ und an diesen Flächen sind die e_1 und e_3 Kurven orthogonale geodätische Linien.

Wenn die Gleitung in einer Ebene entlang nur einer Richtung (z.b. in der von $\mathbf{e}_1$) stattfindet, so liegt der Fall der ebenen Spannung vor. In diesem Falle treten nur Stufenversetzungen und zwar nur parallel zur $\mathbf{x}_3$-Achse auf. Wenn $\mathbf{x}_1$ und $\mathbf{e}_1$ miteinander den Winkel ω einschliessen, so folgt die fundamentale Differentialgleichung

$$(2) \qquad \cos\omega\frac{\partial\omega}{\partial x_2}-\sin\omega\frac{\partial\omega}{\partial x_1}=\frac{\partial\omega}{\partial e_2}=0.$$

Die Lösungen dieser Gleichung werden besprochen. Ein weiteres Problem, dass behandelt wird, ist die Berechnung der Gitterdeformation, wenn die Änderung der Form des festen Körpers gegeben ist. Zuletzt werden noch die bei der ebenen Verbiegung auftretenden allgemeinen Probleme und einige spezielle Beispiele besprochen. *T. Neugebauer* (Budapest)

2137:

Bilby, B. A.; and Gardner, L. R. T. Continuous distributions of dislocations. V. Twisting under conditions of single glide. Proc. Roy. Soc. London. Ser. A. **247** (1958), 92–108.

[Teil IV: siehe #2136.] Wegen des atomistischen Aufbaus der Materie ist selbstverständlich die Annahme kontinuierlich verteilten Versetzungen nur eine mathematische Abstraktion, sie ist jedoch eine sehr brauchbare Näherung. In der vorliegenden Arbeit wird diese Methode auf das Problem der Verdrehung (die Verfasser vermeiden bewusst das Wort "Torsion") eines kreiszylindrischen einachsigen Kristalles angewandt und die Resultate werden mit den experimentellen Ergebnissen und der Theorie von A. D. Whapham und H. Wilman [Proc. Roy. Soc. London Ser. A. 237 (1956), 513–529] verglichen. In der Theorie der Verfasser wird angenommen, dass es sich um nicht zu grosse Verdrehungen pro Längeneinheit und ausserdem bloss um Gleitungen von einer einzigen Art handelt.

$\mathbf{x}_\alpha$ sei ein rechtwinkliges Koordinatensystem, wobei $\mathbf{x}_3$ parallel zur Senkrechten der ursprünglichen Gleitebene orientiert ist und ausserdem sei $\mathbf{e}_a$ ein lokales System in einem Punkte des Kristalles. Dann folgen Zusammenhänge vom Typ

$$\mathbf{e}_a=(e/x)_{a\alpha}\mathbf{x}_\alpha.$$

Sind die Verdrehungen tatsächlich klein, so haben wir

$$(e/x)_{a\alpha}=\delta_{a\alpha}+\varepsilon_{a\alpha\beta}\Phi_\beta{}^L,$$

wo die ε nur gleich ±1 sein können. Die Dichte der Versetzungen wird durch den Torsionstensor $T_{\alpha\beta\gamma}$ beschrieben. Im Falle von kleinen Gitterdeformationen folgt

$$\mathbf{R}_{\alpha\beta}{}^L=-\varepsilon_{\alpha\beta\gamma}\Phi_\alpha{}^L$$

und R hängt durch die Formel $\mathbf{A}_{\alpha\beta}=+\varepsilon_{\beta\gamma\lambda}(\partial R_{\alpha\lambda}{}^L/\partial x_\gamma)$ mit den Burgersschen Vektor $\mathbf{A}_{\alpha\beta}\mathbf{x}_\alpha$ zusammen. Danach wird ein neues Koordinatensystem $\mathbf{y}_\alpha$ eingeführt, dessen $\mathbf{y}_3$-Achse zur Achse des deformierten Kristalls parallel steht. Dann folgt $\Phi_\alpha{}^L\mathbf{x}_\alpha=\Theta_\alpha{}^L\mathbf{y}_\alpha$ und die einzige Grenzbedingung ist jetzt $\Theta_3{}^L=\Theta_3{}^L(y_3)$. Weiter erhält man aus den ursprünglichen Bedingungsgleichungen, dass

$$\Theta_1{}^L=-ky_1+\mu(y_2y_3)$$

sein muss, wobei μ der hyperbolischen Differentialgleichung

$$-\cos\chi_0\frac{\partial\mu}{\partial y_2}+\sin\chi_0\frac{\partial\mu}{\partial y_3}=0$$

genügt. Ausserdem werden noch die Funktionen ψ_1 und ψ_3 eingeführt die mit den von den Verzetzungen herrührenden Deformationen zusammenhängen und die man aus Integralgleichungen berechnet.

Die Resultate der Theorie sind in guter Übereinstimmung mit den erwähnten Experimenten, solange die Verdrehungen pro Längeneinheit nicht zu gross werden, ausserdem stimmen die Ergebnisse dieser vereinfachten Theorie auch weitgehend mit der viel verwickelteren Theorie von Whapham und Wilman überein. Abweichungen werden besprochen. *T. Neugebauer* (Budapest)

FLUID MECHANICS, ACOUSTICS

See also 1841, 2071, 2130, 2221, 2225, 2226, 2227, 2228, 2229.

2138:

Aymerich, Giuseppe. Moti non permanenti di un gas perfetto con linee di corrente invarianti. Rend. Sem. Fac. Sci. Univ. Cagliari **27** (1957), 27–34.

The velocity potential

$$\Phi(x,y;t)=f(t)\varphi(x,y)$$

preserves the streamlines in isentropic, plane, irrotational, nonstationary flows of a perfect gas. The author deduces the equation

$$(c^2-u_1{}^2)\varphi_{xx}-2u_1u_2\varphi_{xy}+(c^2-u_2{}^2)\varphi_{yy}=2\alpha(u_1{}^2+u_2{}^2+\alpha\varphi),$$

which reduces to the homogeneous equation governing stationary flows when the parameter $\alpha=0$, and treats in detail the case of a one-dimensional flow where $\varphi(x,y)=\varphi(x)$. *C. D. Calsoyas* (Livermore, Calif.)

2139:

Walton, J. Note on a source in a rotating fluid. Quart. J. Mech. Appl. Math. **11** (1958), 208–211.

The flow due to a source on the axis of a fluid possessing solid body rotation has been treated by the principle of maximum mass flow. In this note, the author identifies the maximum velocity achieved by the source with the propagation velocity of long waves along the surface dividing the region of solid body rotation from the cylindrical core. *Y. H. Kuo* (Peking)

2140:

Halton, J. H. Elliptical whirl of flooded journal bearings. Proc. Cambridge Philos. Soc. **54** (1958), 119–127.

A thin film of incompressible viscous liquid is contained between a cylindrical journal of diameter D and a cylindrical shell of diameter $D+2\bar{h}$; one of the cylinders is made to rotate with constant-angular velocity ω and one of the two axes is fixed. The approximate equation of Osborne Reynolds is used to determine the pressure in the liquid film in the four cases that arise. The mechanics of the system is examined in a relative motion of the two axes described as an elliptical whirl in which the distance between the axes is

$$[(\nu_0+\nu_1\cos\omega't)^2+(\nu_2\sin\omega't)^2]^{\frac{1}{2}}\bar{h},$$

where ν_0, ν_1, ν_2 and ω' are constant. Two special cases are

considered in detail: the circular whirl in which $v_0=0$, $v_1=v_2$, and the whirl of small amplitude in which v_1/v_0 and v_2/v_0 are both small. *W. R. Dean* (London)

2141:

Spence, D. A. The lift on a thin aerofoil with a jet-augmented flap. Aero. Quart. **9** (1958), 287–299.

This is the extension of the author's theory of jet-flap airfoils [Proc. Roy. Soc. London Ser. A **238** (1956), 46–68; MR **18**, 529] to the case of an airfoil with deflected flap. Actually the jet issues from the flap hinge to prevent boundary-layer separation on the flap, but if it is directed tangentially along the flap surface it is equivalent theoretically to a jet issuing from the trailing edge, and the theory of the earlier paper applies. The integral equation is satisfied approximately. Numerical results have been obtained on the Deuce computer for a range of jet momentum coefficients and flap chord ratios. Lift and chordwise loading are given. Since there will be a transfer of momentum to the boundary layer, the effective jet momentum at the trailing edge will be less than that issuing from the hinge; this point is discussed briefly.

W. R. Sears (Ithaca, N.Y.)

2142:

***Savic, P.; and Boult, G. T. The fluid flow associated with the impact of liquid drops with solid surfaces.** Heat transfer and fluid mechanics institute, held at California Institute of Technology, Pasadena, Calif., June, 1957, pp. 43–84. Stanford University Press, Stanford, Calif. $7.50.

The fluid flow representing the normal impact of a liquid drop on a solid plate is investigated by applying the potential theory of incompressible fluid. Assuming that the drop remains a spherical sector of increasing radius, the authors calculated the shape of the spreading drop and the pressure distribution over the impact plane. Experiments carried out with high speed camera exhibit the main features predicted by the theory. The combined effects of fluid flow and heat transfer were investigated experimentally with water drops impinging on a hot surface and with molten wax drops on a cold surface.

T. Yao-tsu Wu (Pasadena, Calif.)

2143:

Filatov, A. N. On the dynamic action of a liquid on a cistern with arbitrary longitudinal acceleration. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **21** (1957), 107–111. (Russian)

2144:

Filatov, A. N. On helical motions of a barotropic liquid. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **21** (1957), 97–106. (Russian)

2145:

Naumova, L. G. Application of the method of successive approximations to the problem of N. E. Žukovskiĭ for the motion of a liquid in a shallow channel. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 5, 27–31. (Russian)

2146:

Berman, Abraham S. Laminar flow in an annulus with porous walls. J. Appl. Phys. **29** (1958), 71–75.

The problem considered is that of the steady state flow of an incompressible viscous fluid in an annulus bounded by two concentric, cylindrical, porous walls. The radial component of flow is generated by the radial injection of fluid at one wall and by the fluid withdrawal, at the same rate as that of injection, at the other wall; whereas the axial flow is maintained by a pressure gradient. With the assumption that the circumferential component of flow vanishes, the author was able to obtain an exact solution of the Navier-Stokes equations for this problem. The effects of such radial cross flows on the characteristics of the flow field were examined in detail.

T. Yao-tsu Wu (Pasadena, Calif.)

2147:

Paskalew, G.; und Tschobanow, I. Über die strenge Lösung der Navier-Stokes-Gleichungen bei einer Quellstrecke im Halbraum. Z. Flugwiss. **6** (1958), 199–203.

In a recent investigation, Schmieden and Müller [same Z. **4** (1956), 300–309; MR **18**, 252] studied certain exact solutions of the Navier-Stokes equations which represented flows in a half-space bounded by a plane rigid wall and characterized by sources distributed along a perpendicular axis. These are closed-form solutions expressible essentially as the ratio of two polynomials. Here new cases are found in which Schmieden and Müller's equations can be integrated in closed form; these are drawn from the authors' papers on quadrature solutions of the hypergeometric equation [(Yearbook of Sofia University, **50** (1955/56) and **51** (1956/57) (in Bulgarian)]. The hydrodynamic significance is not discussed here.

W. R. Sears (Ithaca, N.Y.)

2148:

***Emersleben, Otto. Wie hängt bei Parallelströmung zäher Flüssigkeiten die Durchflussmenge von der Gestalt des Querschnitts ab?** Anwendungen der Mathematik, Nr. 3. Universität Greifswald, Greifswald, 1958. 21 pp.

This report summarizes the results for flow through straight pipes with various cross-sections.

Y. H. Kuo (Peking)

2149:

***Emersleben, Otto. Über eine exakt berechnete Parallelströmung zäher Flüssigkeiten zwischen gleichmässig verteilten Zylinderflächen, sowie deren Anwendungen. 2 Aufl.** Anwendungen der Mathematik, Nr. 2. Universität Greifswald, Greifswald, 1957. 17 pp.

The author gives a summary account of various applications of a special Zetafunction to the fluid-flow problems. These include: Flow in the large, such as underground water through sandgrains; flow in the small, such as flow through long molecules; flow through porous media; flow problems in macromolecular chemistry, such as behaviors of macromolecules in solution; and flow through assemblages of spheres. *Y. H. Kuo* (Peking)

2150:

Grohne, Diether; und Manohar, Rampurkar. Über ein Charakteristiken-Differenzenverfahren zur Berechnung laminarer Grenzschichten. Z. Angew. Math. Phys. **9b** (1958), 332–346.

The application of a difference-method, taking proper account of the parabolic nature of the partial differential equations, to laminar boundary layer computation is studied with the help of a simple example. The derivatives with respect to the characteristic variable are replaced by difference expressions, so that a set of ordinary differential equations in the other variable (modified to take account of the asymptotic behaviour of the solution far from the wall) remains to be solved numerically. The method shows insight and discrimination, but even so, it fails near separation, and hence, contributes nothing to two-dimensional, isothermal laminar boundary layer calculation which is not furnished by the vastly simpler

method of Thwaites. It provides a useful model, however, for computational procedures suitable for more general, two-dimensional boundary layer problems for which insufficient information is available yet to establish a Thwaites method. The author's results contribute a little more evidence that the difficulties near separation are connected with the interaction between boundary layer and potential flow. *R. E. Meyer* (Providence, R.I.)

2151:

Diederich, Franklin W.; and Drischler, Joseph A. Effect of spanwise variations in gust intensity on the lift due to atmospheric turbulence. NACA Tech. Note no. 3920 (1957), 56 pp.

The method presented by F. W. Diederich [NACA Tech. Note no. 3910] for calculating the effect of spanwise variations in gust intensity on the statistical characteristics of the response of an airplane to continuous random atmospheric turbulence is used herein to calculate the effect of these variations on the lift directly due to turbulence. Both the horizontal and the vertical components of turbulence for both swept and unswept wings are considered. Several analytic approximations to the correlation functions and power spectra of atmospheric turbulence and several spanwise weighting functions (span loadings) are used in these calculations.

The averaging effect of the span on the lift is shown to be very similar for the normal and the longitudinal components of turbulence, for the various span loadings and turbulence spectra considered herein, and for various angles of sweepback. (From author's summary.)

R. C. DiPrima (Troy, N.Y.)

2152:

Townsend, A. A. The effects of radiative transfer on turbulent flow of a stratified fluid. J. Fluid Mech. 4 (1958), 361–375.

An expression is found for the rate at which temperature fluctuations in a turbulent fluid are destroyed by radiative transfer, local thermodynamic equilibrium being assumed. The dynamical interaction between the temperature and velocity fields is considered in two limiting cases: (a) if the heat is radiated before reabsorption over distances large compared with the scale of the turbulent fluctuations, and (b) if the fluid is nearly opaque. Criteria are found for the maintenance of fully turbulent flow in terms of the mean velocity and temperature gradients and of the rate of radiative cooling. If the radiative effects are weak, it is shown that a sudden collapse of the turbulence occurs when the flux Richardson number R_f is less than one, but if they are strong, then the turbulent intensity approaches zero smoothly as $R_f \to 1$. The significance of this interesting work to atmospheric motions in the upper levels is discussed briefly.

O. M. Phillips (Baltimore, Md.)

2153:

Favre, Alexandre. Equations statistiques des gaz turbulents: énergie cinétique, énergie cinétique du mouvement macroscopique, énergie cinétique de la turbulence. C. R. Acad. Sci. Paris 246 (1958), 2839–2842.

This note is the second part of a development of the mean value equations for a turbulent gas in which any or all of the particle velocity, density, internal energy, body forces, viscosity, and conductivity may be variable and turbulent in the macroscopic scale. Equations for the mean values of the total kinetic energy, of the part due to the mean motion and of the part due to the turbulent motion are given, with physical interpretations of the several terms. *A. A. Townsend* (Cambridge, England)

2154:

Maeder, Paul F.; and Thommen, Hans U. On the boundary layer at perforated walls. Z. Angew. Math. Phys. 9b (1958), 438–453.

It is assumed that turbulent flow occurs along a wall having perforations or slots that are small compared to the boundary-layer thickness but not so small that the theories of flow near porous walls apply. It is then assumed that the mixing that occurs is like that in free-jet mixing; i.e., that the mixing zone grows linearly with distance downstream from the leading edge and the velocity profiles are similar. A mixing length proportional to the width of the zone is adopted to describe the turbulence, and its value is taken from free-jet and wake experience. This theory (I) leads to relationships between the average shear stress at the wall and other quantities. Next, mixing at an open-jet boundary is treated (II) by the same technique; this problem was previously solved by Tollmien [Z. Angew. Math. Mech. 6 (1926), 468–478] and others. Finally, the mean wall shear for a perforated wall is calculated by averaging the contributions of the open-jet parts (from (II)) and the solid-wall parts (from experimental data as function of Reynolds number). Put back into (I), this mean shear stress now determines the width of the mixing zone and other results. Agreement with experiments made at Brown University is good.

W. R. Sears (Ithaca, N.Y.)

2155:

***Kampé de Fériet, Joseph. Problèmes mathématiques de la théorie de la turbulence homogène.** Corso sulla teoria della turbolenza, Vol. 1, pp. 1–104. Centro Internazionale di Matematica Estivo. Libreria Editrice Universitaria Levrotto and Bella, Turin, 1957. viii+339 pp.

This is a review article, written in a leisurely manner and containing an extensive bibliography, in which the author discusses some of the mathematical problems that arise in the study of turbulence. Most of the discussion is concerned with questions relating to the statistical description of the turbulence rather than with the dynamical aspects of the subject. *W. H. Reid* (Providence, R.I.)

2156:

Filippov, G. V. On a turbulent flow in the entrance portions of circular tubes. Ž. Tehn. Fiz. 28 (1958), 1823–1828. (Russian)

2157:

Alekseev, N. I. On a steady stream of a compressible viscous fluid admitting a family of orthogonal planes. Aviacion. Inst. Sergo Ordžonikidze. Trudy Inst. no. 61 (1956), 5–19. (Russian)

Alekseev applies the following definition: a steady compressible viscous flow whose trajectories admit a family of orthogonal planes is a flow which does not possess a transverse circulation. He shows that there are three possible cases of such a flow: a parallel flow, a flow with a parallel field of circulation, and a parallel flow with a parallel field of circulation. The analytical aspects of each of these flows are thoroughly discussed.

M. Z. v. Krzywoblocki (Urbana, Ill.)

2158:

Finn, Robert; and Gilbarg, David. Uniqueness and the force formulas for plane subsonic flows. Trans. Amer. Math. Soc. **88** (1958), 375–379.

Proofs somewhat simpler than existing proofs of the following theorems are given: (i) Not more than one two-dimensional isentropic irrotational subsonic flow about an obstacle can have a given circulation and velocity at infinity; (ii) If the obstacle has a trailing edge, where the direction of the tangent changes discontinuously, then at most one value of the circulation is possible, given the velocity at infinity; (iii) The lift on the obstacle is related to this circulation by the usual Joukowski formula.

M. J. Lighthill (Manchester)

2159:

Helliwell, J. B. An application of the Weber-Orr transform to the problem of transonic flow past a finite wedge in a channel. Proc. Cambridge Philos. Soc. **54** (1958), 391–395.

Il s'agit de l'écoulement autour d'un angle placé symétriquement dans un canal de largeur constante, la longueur des cotés de l'angle étant finie. On suppose que la ligne de courant qui suit un des cotés de l'angle se continue en une ligne de jet à vitesse constante. L'écoulement est partout subsonique, mais l'angle est petit et les vitesses à l'infini amont et à l'infini aval sont voisines de la vitesse critique. On peut donc légitimement faire l'étude dans le cadre de la théorie des petites perturbations d'un écoulement transsonique: on doit chercher la solution d'une équation de Tricomi dans le plan de l'hodographe répondant à des conditions aux limites convenables. Cette résolution est ramenée à celle d'une équation intégrale que l'on peut dans le cas présent résoudre explicitement.

P. Germain (Paris)

2160:

Ryhming, I. Über die instationäre Überschallströmung durch Schaufelgitter mit Rückwirkung. Z. Angew. Math. Mech. **37** (1957), 416–431. (English, French and Russian summaries)

Es wird die Überschallströmung durch zwei aneinander vorbeibewegte Plattengitter gleicher Teilung untersucht. Die Schaufeln des Laufrades sind gegenüber der Zuströmrichtung etwas angestellt. Die Geschwindigkeitskomponenten senkrecht zur Gitterfront sind kleiner als die Schallgeschwindigkeit, so dass vom Laufrad ausgehende Störungen in das Leitrad gelangen können. Die quasistationäre Strömung ist vom gleichen Verfasser bereits in einer früheren Arbeit [dieselbe Z. **37** (1957), 370–385; MR **19**, 914] untersucht worden. In der vorliegenden Arbeit werden die Besonderheiten des instationären Vorganges untersucht. Durch die kurzzeitige Änderung der Reflektion von Stössen, die durch die Relativbewegung der Gitter vorübergehend vor die benachbarte Schaufel gelangen, werden instationäre Wellen ausgelöst. Die Berechnung dieser Wellen wird über eine Lösung des Störpotentials durchgeführt. Die Diskussion des Ergebnisses zeigt, dass die instationäre Wellenbewegung den durch eine quasistationäre Betrachtung ermittelten Strömungsverlauf nur unwesentlich ändert.

L. Speidel (Mülheim)

2161:

Elrod, Harold G., Jr. Note on a solution of the telegraphist's equation applicable to supersonic shear flow. J. Math. Phys. **37** (1958), 66–68.

The telegraphist's equation $\partial^2\nu/\partial x^2-\partial^2\nu/\partial\sigma^2=-\lambda^2\nu$, under the boundary conditions $\nu=0$, $x-\sigma\leq 0$, $\sigma\geq 0$, and $\nu=g(x)$, $x>0$, $\sigma=0$, was solved by C. Chang and J. Werner [J. Math. Phys. **31** (1952), 91–101; MR **14**, 178]. Their solution involved $\partial\nu/\partial\sigma(x, 0)$. In order to determine this unknown function, they obtained an integral equation for it, which was then solved in terms of an infinite series of Bessel functions. The purpose of the present note is to show that this integral equation can be solved much more simply by the use of Laplace transforms.

J. Elliott (New York, N.Y.)

2162:

Chang, Che-Tyan. Interaction of a plane shock and oblique plane disturbances with special reference to entropy waves. J. Aero. Sci. **24** (1957), 675–682.

Consider a steady motion consisting of two regions of uniform flow separated by a plane shock wave. Suppose the velocity, entropy, etc. experience small time-dependent perturbations $\nabla\phi+\nabla\times\bar{E}$, S, etc., and that the shock wave is also slightly deformed. Linearize the partial differential equations of non-steady flow by retaining only terms of the first order in ϕ, $\bar{E}$, and S, and similarly simplify the shock conditions. Eventually this yields six homogeneous differential equations, each of which involves only one of ϕ, $\bar{E}$, or S. The author chooses in the region ahead of the shock solutions $\phi=0$, $\bar{E}=0$, $S_1=R_s\cos k_1(l_1U_1t-l_1x_1-m_1y_1)$, where l_1, m_1 is a constant unit vector, and R_s, k_1, U_1 are constants. In the region on the downstream side of the shock ϕ and $\bar{E}$ are non-constant and the type of the equation for ϕ depends on the orientation of l_1, m_1. The author summarizes his work for a particular normal shock and reports 1) for small enough $|m_1/l_1|$ the phases, at the shock, of the downstream waves are the same as for S_1, and the perturbed sound waves have constant form; 2) for large $|m_1/l_1|$ the phases at the shock change, and the sound waves attenuate.

J. H. Giese (Aberdeen, Md.)

2163:

Chang, C. T. On unsteady interaction between a weak thermal layer and a strong plane oblique shock. J. Aero. Sci. **25** (1958), 317–323.

In the convex region bounded by two straight walls consider a plane motion consisting of two regions of steady uniform flow separated by a plane shock through the corner. Perturb this flow by introducing ahead of the shock a contact discontinuity which advances at constant speed and at which there is a small change of temperature. To find the non-steady perturbed motion behind the shock the author seeks for the equations developed in the preceding paper self-similar solutions in which velocity components, pressure, etc. are functions of x/t and y/t only. Eventually this yields a partial differential equation of mixed type for the pressure perturbation. In the hyperbolic region adjacent to the corner there are only regions of uniform flow. In the elliptic region there arises a mixed boundary value problem for Laplace's equation, solved by a method employed by M. J. Lighthill [Proc. Roy. Soc. London Ser. A **198** (1949) 454–470; MR **11**, 478] to investigate diffraction of plane blast waves by wedges.

J. H. Giese (Aberdeen, Md.)

2164:

Carrier, G. F. Shock waves in a dusty gas. J. Fluid Mech. **4** (1958), 376–382.

The problem attacked is that of a steady, plane, normal shock in a uniform flow of a perfect gas carrying suspended dust particles. The partial pressure of the dust is neglected, and it is postulated that no evaporation of the dust particles occurs. It is assumed that the shock front con-

sists of a classical gas shock which does not immediately affect the dust particles. The action of the dust is to transfer momentum to and accept heat from the gas behind the front. Far downstream the gas and dust must have the same temperatures and velocities. This asymptotic state is calculated from the conservation laws; it differs increasingly from the state just behind the gas shock with increasing values of the ratio of heat capacity of the dust per unit volume to that of the gas. To determine conditions between these two states, one must postulate the mechanisms of momentum and heat transfer. Here it is assumed that these are characterized by a particle drag coefficient and Nusselt number whose ratio is proportional to the particle's Reynolds number. This leads to an ordinary differential equation, some integral curves of which are reproduced here. In lieu of exhaustive results the dimensionless forms of the equations, suitable for further numerical computations, are presented.

W. R. Sears (Ithaca, N.Y.)

2165:

Westervelt, Peter J. Scattering of sound by sound. J. Acoust. Soc. Amer. **29** (1957), 934–935.

The result of a previous paper [same J. **29** (1957), 199–203; MR **19**, 915], that no second-order scattered sound arises from the interaction of a perfectly collimated plane-wave beam of sound crossing another one at an angle of $\frac{1}{2}\pi$, is extended to the case of arbitrary angle of crossing, with some additional discussion.

M. J. Lighthill (Manchester)

2166:

Lyamšev, L. M. Sound diffraction on an infinite thin elastic cylindrical shell. Akust. Ž. **4** (1958), 161–167. (Russian)

Using Kennard's theory [J. Appl. Mech. **20** (1953), 33–40; MR **14**, 817] for shell vibrations, together with Green's theorem and Sommerfeld's radiation condition, the scattering of plane acoustic waves at oblique incidence on an infinitely long, thin, deformable cylindrical shell is studied with the aid of an integro-differential equation method described earlier by the author (1955). The solution is reduced to that of a set of linear algebraic equations when the particle velocities and the combined incident plus known scattered field pressure of a rigid cylinder are expressed in trigonometric series form.

W. W. Soroka (Berkeley, Calif.)

2167:

Prokof'ev, V. A. The part played by radiation in the hydrodynamic theory of the propagation of plane forced waves of infinitely small amplitude. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 6, 7–16. (Russian)

Disagreeing with previous formulations for the radiation of heat in sound wave propagation, the author makes use of a linearized Navier-Stokes equation, an equation of radiant heat transfer due to Unzold (1949), Kirchhoff's fourth power law, and a forced simple harmonic plane wave disturbance at $x=0$ to show that waves of three categories are developed, namely, attenuated acoustic waves, attenuated waves of thermal conduction, and attenuated waves of thermal radiation. Propagation characteristics of these waves are developed.

W. W. Soroka (Berkeley, Calif.)

2168:

Nyborg, Wesley L. Acoustic streaming near a boundary. J. Acoust. Soc. Amer. **30** (1958), 329–339.

Following Schlichting (1932), the flow in a thin viscous layer next to a solid boundary is considered largely independent of the outer flow field. An approximate solution for boundary layer flow is obtained by adding to the known oscillatory irrotational velocity distribution assumed velocity distributions which satisfy approximately Lamb's equation for oscillatory flow in a viscous medium and which help enforce the boundary conditions at the given surface. The earlier-developed streaming equations [Nyborg, same J. **25** (1953), 938–944; MR **15**, 265] are then solved. Certain limitations apply to the kinematic viscosity-frequency ratio. Resonant gas bubbles on submerged surfaces are shown to result in strong microstreaming.

W. W. Soroka (Berkeley, Calif.)

2169:

Slezkin, N. A. On the flow of a viscous liquid with a free boundary on a porous base. Vestnik. Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 5, 3–5. (Russian)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 1836, 1837, 2210, 2228, 2229.

2170:

Toraldo di Francia, Giuliano. Il problema matematico del sistema ottico concentrico stigmatico. Ann. Mat. Pura Appl. (4) **44** (1957), 35–44.

The author gives a mathematical solution of the trajectories in an inhomogeneous optical space in which a sphere is sharply imaged on a sphere, expressing r (radius vector) and n (refractive index) in parametrical form, giving a numerical solution for a particular case.

M. Herzberger (Rochester, N.Y.)

2171:

Bonštedt, B. È. Calculation of the electrostatic field of a diaphragm system. Ž. Tehn. Fiz. **28** (1958), 1801–1808. (Russian)

2172:

Balazs, N. L. Effect of a gravitational field, due to a rotating body, on the plane of polarization of an electromagnetic wave. Phys. Rev. (2) **110** (1958), 236–239.

It is shown that for electromagnetic waves the gravitational field of a rotating body acts as an optically active medium. Thus the plane of polarization of the wave rotates while it passes through this field. The effect is small. The angle of rotation due to the gravitational field of the sum is about 10^{-12} radian. (Author's summary).

V. M. Papadopoulos (Providence, R.I.)

2173:

Totaro, Carmelo. Su un problema al contorno della elettrodinamica dei corpi in moto. Boll. Un. Mat. Ital. (3) **12** (1957), 658–663.

L'A. studia i fenomeni della riflessione e della rifrazione delle onde elettromagnetiche nel caso di mezzi in moto muovendo dalle condizioni al contorno per le equazioni di Minkowski.

La ricerca riguarda precisamente il caso di due mezzi dielletrici, isotropi ed elettricamente distinti, di cui uno sia fluido. Se il mezzo 2 si suppone sede di un campo primario e si indica con 1 l'altro mezzo, si ammette che il mezzo 2 sia in quiete rispetto all'osservatore e il mezzo 1 in moto traslatorio rettilineo uniforme rispetto al mezzo 2 con velocità normale alla superficie di separazione dei due mezzi.

G. Lampariello (Roma)

2174:

Manarini, Anna Marisa. Considerazioni sulla propagazione guidata e libera delle onde elettromagnetiche nei mezzi in moto. Rend. Sem. Mat. Univ. Padova **27** (1957), 60–74.

L'A. ritrova per altra via alcuni risultati sulla propagazione di onde elettromagnetiche nei conduttori in moto o in una guida di onde percorsa da un dielettrico fluido (in moto traslatorio rettilineo uniforme) già conseguiti da A. Ferla [Boll. Un. Mat. Ital. (3) **11** (1956), 227–237; MR **18**, 357], T. Zeuli [ibidem, 189–197; MR **18**, 357], C. Agostinelli [Univ. e. Politec. Torino Rend. Sem. Mat. **14** (1954–55), 257–268; MR **17**, 808].

G. Lampariello (Roma)

2175:

Hižnyak, N. A. Green's function of Maxwell equations for inhomogeneous media. Ž. Tehn. Fiz. **28** (1958), 1592–1609. (Russian)

By means of a Green's function technique a pair of integral equations is obtained which determines the electromagnetic field in the presence of a gyrotropic body of arbitrary tensor dielectric constant and permeability excited by an arbitrary electromagnetic field. As an example, the scattering of an electromagnetic wave by a small anisotropic ellipsoid and a thin anisotropic elliptic cylinder is calculated. Also it is shown that the anisotropic dielectric ellipsoid and elliptic cylinder are the only convex bodies for which the electrostatic fields within and without can be homogeneous.

C. H. Papas (Pasadena, Calif.)

2176:

Krumin', Yu. K. A problem of a conducting sphere in a moving magnetic field. Latvijas PSR Zinātņu Akad. Vēstis **1957**, no. 5(118), 119–129. (Russian. Latvian summary)

A spherical conductor is symmetrically placed in a cylindrical tube (axis along z) on the surface of which a magnetic field varying periodically with time and z is present. The induced magnetic field in the conductor is determined and the force acting on it is calculated.

S. Chandrasekhar (Williams Bay, Wis.)

2177:

Cole, C. F., Jr. The characteristics of an electromagnetic wave reflected from a moving object. J. Franklin Inst. **265** (1958), 463–471.

A theoretical study is carried out to investigate the difference in the properties of an electromagnetic wave reflected from a fixed object and a moving object. Solutions of a wave equation in rectangular coordinates are used with different boundary conditions for the determination of the reflected wave characteristics.

The variation in the values of angle of incidence and reflection with time are shown to be related to the Doppler frequency as well as the relative velocity. Expressions for an electromagnetic wave reflected from both the fixed and moving object are given. A relationship between relative velocity with respect to the source of incident wave and the reflected wave frequency and amplitude is described. (Author's abstract.)

V. M. Papadopoulos (Providence, R.I.)

2178:

Kodis, Ralph D. Variational principles in high-frequency scattering. Proc. Cambridge Philos. Soc. **54** (1958), 512–529.

A pair of variational principles are formulated for two-dimensional scattering by obstacles. The first of these is in terms of the obstacle boundary values, and it is shown that a simple 'optical' trial function leads to an incorrect frequency dependence for the scattering cross-section. In the second, the obstacle is viewed as the analogue of an aperture coupling two half spaces. The geometric optics part of the cross-section can then be made explicit and is split off to leave a stationary form for the frequency correction. The zero-order calculation for the cross-section of a circle, using corresponding 'optical' trial functions, is found to have the correct $(ka)^{-\frac{1}{3}}$ frequency dependence. (Author's abstract.)

E. T. Copson (Fife)

2179:

Smirnov, N. N. Propagation of electromagnetic waves in circular waveguides with periodic slits. Ž. Tehn. Fiz. **28** (1958), 1494–1504. (Russian)

The propagation parameters of electromagnetic waves in circular waveguides with periodic annular slots or a single spiral slot are calculated.

C. H. Papas (Pasadena, Calif.)

2180:

Bogdankevič, L. S. Motion of a charged particle in a rectangular waveguide filled with an anisotropic dielectric. Ž. Tehn. Fiz. **28** (1958), 1505–1509. (Russian)

In this paper the author finds the loss in energy of a charged particle moving along the axis of a rectangular waveguide filled with a uniaxial crystal oriented either parallel or perpendicular to the axis of the waveguide.

C. H. Papas (Pasadena, Calif.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 2152, 2221.

2181:

Vernotte, Pierre. Les paradoxes de la théorie continue de l'équation de la chaleur. C. R. Acad. Sci. Paris **246** (1958), 3154–3155.

It is known that Fourier's equation of heat conduction which is parabolic, implies an infinite velocity propagation, or, in other words, that the mechanism of heat conduction is established instantaneously under all conditions. This is unacceptable on physical grounds in spite of the fact that Fourier's law agrees well with experiment. However, discrepancies are likely to occur when extremely short distances or extremely short time intervals are considered, as they must in some modern problems of aero-thermodynamics.

In order to remove this difficulty the author proposes to replace Fourier's assumption by

(a) $$-\lambda T_x=\phi+\tau\phi_t,$$

where λ is the conductivity and ϕ is the heat flux. The quantity τ is a very small characteristic time and for $\tau\to 0$ (a) reduces to Fourier's law. The resulting equation for the temperature field becomes

(b) $$aT_{xx}-\tau T_{tt}-T_t=0$$

(a=diffusivity) and is hyperbolic; hence it possesses real characteristics and implies a finite velocity of propagation of order $(a/\tau)^{\frac{1}{2}}$, or about 150 m/sec for nitrogen.

The applicability of (a) to real systems can only be judged with reference to experiments or by the methods of statistical mechanics.

J. Kestin (London)

2182:

Cattaneo, Carlo. Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanée. C. R. Acad. Sci. Paris **247** (1958), 431–433.

Referring to the paper reviewed above, the author draws attention to his treatment of the same problem published ten years earlier [Atti Sem. Mat. Fis. Univ. Modena **3** (1948), 3–21]. He arrived at substantially the same result by an elementary argument based on kinetic theory and showed that the velocity of propagation is

$$V=\sqrt{(\lambda^2/\sigma c_v\rho)}=(1/\sqrt{3})c\overline{\lambda_c}/\sqrt{(\overline{l^2})},$$

where c is the particle velocity (in a gas) λ_c is the mean free path for a molecule travelling with the velocity c, $\sigma=\frac{1}{2}N\propto\overline{l^2}$; N is the number of molecules per unit volume, $\propto$ denotes a constant, and $\overline{l^2}$ is the usual mean free path.

It is clear that the problem discussed in this and in the preceding paper merits a good deal more attention than could be devoted to it in these two brief notes.

J. Kestin (London)

2183:

Eĭdel'man, S. D. Some linear problems of the theory of heat conduction. Černivec. Derž. Univ. Nauk. Zap. Ser. Fiz.-Mat. **19** (1956), no. 4, 97–108. (Ukrainian. Russian summary)

2184:

Drahlin, E. H. The solution of the equations for a case of stationary heat convection in an infinite circular cylinder. Prikl. Mat. Meh. **21** (1957), 693–695. (Russian)

2185:

Wilson, J. C. Heat transfer at rectangular corners. Quart. Appl. Math. **15** (1957), 195–198.

The two-dimensional heat-conduction equation is solved in a rectangle $0<x<a$, $0<y<a$, with zero temperature on the sides $x=a$ and $y=a$, temperature $k(1-x^2/a^2)$ on the side $y=0$, temperature $k(1-y^2/a^2)$ on the side $x=0$, and zero temperature initially.

M. J. Lighthill (Manchester)

2186:

Klátil, Jiří. Champ de température stationnaire dans une plaque plane infinie à plusieurs couches, avec une distribution arbitraire de la température aux faces bordantes. Apl. Mat. **2** (1957), 258–278. (Czech. Russian and French summaries)

The author deduces formulas which express the stationary temperature in an infinite slab with n layers in the case that the two outside layers are kept at an arbitrary but fixed temperature distribution. The present paper is a generalization of a previous result by the same author [Czechoslovak J. Phys. **6** (1956), 551–557] concerned with only two layers.

František Wolf (Berkeley, Calif.)

2187:

Šikin, I. S. Investigation of certain problems of detonation and burning in media with variable density. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 4, 49–59. (Russian)

2188:

Mička, Jiří; and Schmidt, Oskar. Empirische Formeln, die in den physikalisch-chemischen Applikationen vorkommen. Apl. Mat. **2** (1957), 469–478. (Czech. Russian and German summaries)

Diese Arbeit schliesst, dem Inhalt und der Lösungsweise nach, an Apl. Mat. **2** (1957), 133–153 [MR **19**, 684]. Es sind weitere Typen, die häufig in den physikalisch-chemischen Aplikationen vorkommen, verarbeitet. Die gemessenen Werte können beliebig tabelliert werden, bloss die Abhängigkeiten, die vier Konstanten enthalten, müssen die Bedingung $x_1+x_2=x_3+x_4=k$ erfüllen. Der zweite Abschnitt enthält die mathematische Verifikation der beschriebenen Methode. Im letzten Abschnitte sind einige Beispiele angegeben.

Zusammenfassung der Autoren

QUANTUM MECHANICS

See also 2219.

2189:

Zaĭkov, Raško. Symmetrische Form der Nukleonengleichung. Izvestiya Bulgar. Akad. Nauk. Otd. Fiz.-Mat. Tehn. Nauk Ser. Fiz. **6** (1957), 3–11. (Bulgarian. Russian and German summaries)

A generalization of the Dirac equation for nucleons is proposed in which the wave function has eight components (of which the first four describe charged, and the second four uncharged, nucleons) but there are three coordinates in addition to those of space-time. The equation is

$$\{\Gamma_jD_j+A_0I\}\omega=0,$$

$$D_j=\partial_j-\frac{e}{\hbar}\{i\phi_j+\tfrac{1}{3}\Gamma_k\Gamma_l\phi_{jkl}+\tfrac{1}{120}\Gamma_k\Gamma_l\Gamma_m\Gamma_n\phi_{jklmn}\},\quad e=\varepsilon/c,$$

where the Γ_j are hermitean matrices satisfying

$$\Gamma_j\Gamma_k+\Gamma_k\Gamma_j=\delta_{jk}\ (j,k=1,\cdots,7),$$

the ϕ_j, ϕ_{jkl}, ϕ_{jklmn} are arbitrary functions of an isotopic three vector $\mathbf{k}$ and a certain number of space-time field functions and their first derivatives (π-meson field, electromagnetic fields, etc.). It is assumed that

$$w=(x_1x_2x_3x_4)\exp i\mathbf{k}\cdot\boldsymbol{\xi},$$

where $\xi=(x_5, x_6, x_7)$. The physical interpretation of quantities of the form $\omega^+A\omega$ is discussed for A certain products of the Γ's. *A. S. Wightman* (Princeton, N.J.)

2190:

Schönberg, M. Quantum kinematics and geometry. Nuovo Cimento (10) **6** (1957), supplemento, 356–380.

The author argues: the fact that using Planck's constant and the velocity of light one can reduce the dimensions of all physical quantities to powers of a length "...indicates the existence of a deep unity between physics and geometry non-equivalent to that of the general theory of relativity." He is devoting a series of papers to exploring this connection, of which the present constitutes an extended preview (mainly without proofs.) The paper contains such a rich variety of geometric analogies and points of view that only a sample can be listed here (for more details see the reviews of the following papers). To each n-dimensional vector space over the reals and its dual there are associated two algebras G_n and L_n which are associated respectively with pseudo-orthogonal geometry and symplectic geometry in a space of $2n$ dimensions. Contained as subalgebras of G_n are the Duffin-Kemmer-Petiau algebras which occur in the theory of integer spin particles and the Clifford algebra which occurs in the theory of half odd integer spin particles.

When one lets $n\to\infty$, the elements of G_n and L_n become naturally related to amplitudes describing states in a second quantized theory with Fermi and Bose statistics respectively. By passing from algebras over the real numbers to algebras over the complex one acquires extra degrees of freedom which may be interpreted as describing different kinds of particles. Since in this geometric approach there is a natural way to construct spinors out of tensors the method leads to the usual relativistic wave equations for free particles of spin $\frac{1}{2}$ as well as those of spin 0 and 1. *A. S. Wightman* (Princeton, N.J.)

2191:

Schönberg, Mario. Quantum mechanics and geometry. An. Acad. Brasil. Ci. **29** (1957), 473–499.

This paper commences the detailed exposition of the ideas described in the paper reviewed above. The author associates with the pair consisting of an n dimensional vector space V_n (over the real numbers) and its dual V_n', the Clifford algebra G_n of a $2n$ dimensional vector space S_{2n}, with a metric of n pluses and n minuses. The correspondence is obtained by writing $S_{2n}=V_n'\oplus V_n$. If $z_j=x_j\oplus y_j$ with $x_j\in V_n'$ and $y_j\in V_n$, then the metric in S_{2n} is given by $(z_1, z_2)_+=(x_1, y_2)+(x_2, y_1)$ and the elements of the Clifford algebra (z_j) corresponding to the vectors z_j satisfy

$$(z_1)(z_2)+(z_2)(z_1)=2(z_1, z_2)_+.$$

The automorphisms of V_n induce automorphisms of V_n' and these together leave $(\ ,\ )_+$ invariant so that the general linear group of V_n can be regarded as a subgroup of the pseudo-orthogonal group of S_{2n}. There is a second algebra, L_n, which is associated with S_{2n}. One introduces the metric $(z_1, z_2)_-=(x_1, y_2)-(x_2, y_1)$. L_n is generated by elements $\{z\}$ (corresponding to the vectors z) which satisfy

$$\{z_1\}\{z_2\}-\{z_2\}\{z_1\}=(z_1, z_2)_-.$$

In analogy with the case of the metric $(\ ,\ )_+$, the general linear group is here regarded as a subgroup of the symplectic group of S_{2n}. Contained in G_n are Duffin-Kemmer-Petiau subalgebras. To obtain them one chooses dual bases in V_n and V_n' which respectively determine elements (I^j) and (I_j), $j=1, \cdots, n$, of L_n. Then, defining

$$(P^{[\nu\cdots]}_{k_1\cdots k_\rho})=(I^{j_1})\cdots(I^{j_\rho})(P)(I_{k_1})\cdots(I_{k_\rho}),$$
$$(P)=(I_1)\cdots(I_n)(I^n)\cdots(I^1),$$
$$(\pi_r)=\frac{1}{p!}\sum_j (P^{[\nu\cdots]}_{k_1\cdots k_\rho}),$$
$$\beta_j{}^{(p)}=(\pi_p)(I_j)+g_{jk}(I^k)(\pi_p),$$

one has

$$\beta_j{}^{(p)}\beta_k{}^{(p)}\beta_l{}^{(p)}+\beta_l{}^{(p)}\beta_k{}^{(p)}\beta_j{}^{(p)}=g_{jk}\beta_l{}^{(p)}+g_{kl}\beta_j{}^{(p)},$$

the defining relation for the generators of a D-K-P algebra. All irreducible D-K-P algebras can be obtained by this or related methods. All automorphisms of even dimensional Clifford algebras are inner, and the effect of several special automorphisms of G_n on its constituent D-K-P algebras is computed. The author describes other algebras associated with the points of V_n (i.e. elements of the algebra are associated with the homogeneous space of translations of V_n). For example, define $\{P\}$ just as (P) above except with curly brackets. Then the automorphism $\Lambda\to\exp(-\{y\})\Lambda\exp(\{y\})$ with $\Lambda\in L_n$, $y\in V_n'$ leaves $\{x\}$ invariant for $x\in V_n$, transforms $\{P\}$ into $\{P\}\exp\{y\}$ and $\{u\}$ into $\{u\}-(u, y)_-1$. $\{P\}$ is then associated with the origin of coordinates in V_n and $\{P\}\exp\{y\}$ to the point at the end of the vector $y\in V_n$, obtained from the origin by the above translation automorphism. This algebra of points is essentially different from that of Grassmann. *A. S. Wightman* (Princeton, N.J.)

2192:

Vachaspati. Transformations in the functional space of quantum mechanics. Proc. Indian Acad. Sci. Sect. A. **47** (1958), 179–183.

The author earlier discussed a generalization of the formalism of quantum mechanics for a system with the finite dimensional Hilbert space of states in which a formal analogy was set up between the normalization condition for a state, and that for a four velocity of a particle in the classical theory of point particles [Mat.-Fys. Medd. Danske Vid. Selsk. **30** (1956), no. 21; MR **18**, 851]. Here ψ^m are complex numbers, the coefficients of the expansion of a state ψ in some orthonormal set. The possibility was discussed that $\psi^m=dx^m/dt$ where the x^m are "coordinates", and the usual quantum mechanics was generalized to yield a Riemannian geometry in the x^m. Here an analogous case is treated where the m is a continuous label. The author considers the problem of defining point transformations of such a system which are not linear. He finds none and therefore concludes it is impossible to define any "Riemannian geometry". To the reviewer the problem of this paper seems ill posed, and according to any reasonable interpretation, the solution incorrect. *A. S. Wightman* (Princeton, N.J.)

2193:

DeWitt, Bryce S. Dynamical theory in curved spaces. I. A review of the classical and quantum action principles. Rev. Mod. Phys. **29** (1957), 377–397.

The paper is a "study··· of a number of usually neglected aspects of standard quantum mechanics which must be considered when attention is focussed on the non-linear features arising from the metric structure of the configuration space of a certain general class of systems." The paper commences with a review (elementary and purely formal) of the transformation theory of classical Hamiltonian systems. There follows a treatment of the analogy between classical Poisson brackets and quantum mechanical commutators. (Again the treatment is purely formal, such delicacies as those discussed by Van Hove [Mem. Acad. Roy. Belgique Cl. Sci. Mem. Coll. in 8° **26** (1951), no. 6; MR **15**, 198] being ignored.) The decisive assumption is then made that the coordinate configuration space has a natural metric. This forces the appearance of various metric dependent terms in the Schrödinger equation. The formalism is applied to the case of a non-relativistic particle moving in a curved space of n dimensions. In the closing section, the effect on the Feynman history integrals of the appearance of metric dependent terms is considered, and certain ambiguities are found in the transition from classical to quantum theory (the ambiguous terms in the Hamiltonian vanish both in the classical limit and in the case of flat space). *A. S. Wightman* (Princeton, N.J.)

2194:

Lomsadze, Yu. M. On an attempt at generalizing contemporary quantum theory. Užgorod. Gos. Univ. Naučn. Zap. **18** (1957), 95–100. (Russian)

The author proposes a method of constructing non-local quantum theories and applies it to the free electron and free photon. The idea is to work with field quantities integrated over space time regions σ whose volume is constrained to be larger than λ^4 where λ is a "fundamental

length". (Actually, most of the discussion is in terms of wave functions of single free particles rather than fields but it applies equally well for fields.) For example, he writes

$$\Psi(\sigma)=\frac{1}{\lambda^4}\int_\sigma \Psi(x)dx,\ \lambda^4=\int_\sigma dx$$

and for the Dirac equation

$$\left(\gamma_\mu \frac{\partial}{\partial\sigma_{x_\mu}}+\varkappa_0\right)\Psi(\sigma)=0$$

where

$$\frac{\partial}{\partial\sigma_{x_\mu}}\Psi(\sigma)=\lim_{\Delta x\to 0}\frac{1}{\Delta x}[\Psi(\sigma')-\Psi(\sigma)]$$

and σ' is obtained from σ by a translation in the μ direction of magnitude x. {Reviewer's remark: Without the constraint on the size of the regions this formalism is a special case of that customary in quantum field theory in which one works with the "smeared field quantities" (operator valued distributions). See, for example, W. Schmidt and K. Baumann, Nuovo Cimento (10) **4** (1956), 860–886 [MR **19**, 97]. There is good evidence from perturbation theory that in case of coupled fields it is essential to use test functions which have a large number of derivatives rather than characteristic functions of sets used by the author.} The case of coupled fields is left open.

A. S. Wightman (Princeton, N.J.)

2195:

*Kemble, Edwin C. **The fundamental principles of quantum mechanics with elementary applications.** Dover Publications, Inc., New York, 1958. xviii+611 pp. $2.95.

The re-issue of this useful text [1st ed.; McGraw-Hill, New York and London, 1937] in an inexpensive format is an event to be welcomed by students and teachers of quantum mechanics. As is indicated in the title, it deals primarily with the foundations of the subject rather than with special problems. Only the non-relativistic theory is treated. Considerable emphasis is placed on the statistical interpretation of the theory, and on the concept of measurement as its intuitive basis. The formulation derives from von Neumann's analysis, of course, but has the flavor of an original essay.

It is natural that some parts of the text have been outdated by developments during the two decades since its first appearance. The eigenfunction problem (in L^2-space) has been explored in masterly fashion by E. C. Titchmarsh [Eigenfunction expansions associated with second-order differential equations, Clarendon Press, Oxford, vol. 1, 1946; vol 2, 1958; MR **8**, 458; **20** #1065]. The approach to the scattering problem via the eigenfunction expansion method has proved to be so difficult as to be ineffective in practice, although this is undoubtedly a matter of technique rather than of principle. In any event, a more abstract approach [J. M. Cook, J. Math. Phys. **36** (1957), 82–87; MR **19**, 1011; and J. M. Jauch, Helv. Phys. Acta **31** (1958), 127–158; MR **20** #682] seems to be definitely more promising. The question of the adequacy of L^2-space as a mathematical model for quantum mechanics remains undecided.

E. L. Hill (Minneapolis, Minn.)

2196:

Bardeen, J.; Cooper, L. N.; and Schrieffer, J. R. Theory of superconductivity. Phys. Rev. (2) **108** (1957), 1175–1204.

This very important paper presents a theory of superconductivity based on a simplified form for the hamiltonian of the conduction electrons and a very ingenious "Ansatz" for the wave functions of ground and excited states. In addition to the screened Coulomb interaction, the hamiltonian contains an electron-electron interaction resulting from the well known electron-lattice coupling. A highly simplified form is adopted for these interactions and, on the basis of clearly presented arguments, a major approximation is made, essentially restricting the interaction term in the hamiltonian to those pairs of electrons which have opposite momenta and spins. A special form is postulated for all eigenstates of the system, involving electron pairs of two types ("ground" pairs and "excited" pairs) as well as single excited electrons. The functions defining these single particle and pair states are determined by a variational method, both at zero and positive temperatures T. For all T below a critical temperature T_c, the excited electrons (as they are defined in the approximate scheme of the authors) are found to have an energy gap of the type required to account for superconductivity. Numerically, the results are also quite satisfactory. The authors calculate single electron matrix elements of the type required for the interpretation of many effects. Electrodynamic properties are investigated in the model, with the result that a Meissner effect is predicted and that satisfactory values are obtained for field penetration depths. In short, although based on a severely truncated hamiltonian, this theory achieves an impressive amount of success in accounting for the equilibrium properties of superconductors.

L. Van Hove (Utrecht)

2197:

Schafroth, M. R. Remarks on the Meissner effect. Phys. Rev. (2) **111** (1958), 72–74.

Earlier work by the author has shown that the theory of the Meissner effect (exclusion of the magnetic field in superconductivity) must take explicit account of gauge invariance, essentially because it is gauge invariance which prevents it in normal materials. This paper suggests that the Bardeen-Cooper-Schrieffer theory [see papers reviewed above] has not done so, and that the demonstration of gauge invariance by the reviewer [see paper reviewed below] is not mathematically complete, both of which are true; but it then suggests that these theories have not answered the main physical difficulties, which is not true. Work in press by Pines and Schrieffer and by Anderson is forthcoming which further completes the Meissner effect demonstration.

P. W. Anderson (Murray Hill, N.J.)

2198a:

Anderson, P. W. Coherent excited states in the theory of superconductivity; Gauge invariance and the Meissner effect. Phys. Rev. (2) **110** (1958), 827–835.

2198b:

Anderson, P. W. New method in the theory of superconductivity. Phys. Rev. (2) **110** (1958), 985–986.

Experiments have shown that the phenomenon of superconductivity in metals has its origin in the interaction between lattice vibrations and electrons. Attempts to formulate this fact in terms of quantum mechanical theory based on the use of simple approximate interaction operators have proved difficult, owing to the complexity of the problem and the troubles of perturbation theory. The author's purpose is to show that objections which have been raised on general theoretical grounds against the theory proposed by Bardeen, Cooper, and Schrieffer

[see last but one review above] can be surmounted so that it provides a valid explanation of the magnetic effects of superconductors (Meissner effect). The main argument is that the terms in the Hamiltonian operator used by Bardeen et al. which are not gauge-invariant have a negligible effect for weak coupling. The author indicates that this theory has been developed, in work as yet unpublished, to give an extension of that of Bardeen et al.

E. L. Hill (Minneapolis, Minn.)

2199:

Born, M.; und Ludwig, W. Zur Quantenmechanik des kräftefreien Teilchens. Z. Physik **150** (1957), 106–117 (1958).

This paper studies in detail the motion of a particle moving freely in a finite one-dimensional interval with perfectly reflecting ends. The time-dependent wave function is written in various forms exhibiting the particle-like or wave-line aspects of the motion.

L. Van Hove (Utrecht)

2200:

Kar, K. C.; and Paria, B. N. A new derivation of Klein-Nishina formula without matrices. Indian J. Theoret. Phys. **5** (1957), 51–62.

2201:

***Edmonds, A. R. Angular momentum in quantum mechanics.** Investigations in Physics, Vol. 4. Princeton University Press, Princeton, N. J., 1957. viii+146 pp. $3.75.

The theory of angular momentum in quantum mechanics is subsumed mathematically under the theory of representations of the 3-dimensional rotation group, the structure of which is well known. [See, e.g., H. Weyl, Theory of groups in quantum mechanics, Dover, New York, 1931.] However, for applications, particularly to nuclear physics, one needs the explicit matrix elements of the unitary transformations which reduce the multiple direct products of irreducible representations, or more often, certain sums over these matrix elements. Most of the progress in evaluating such expressions has been due to the development of the algebra of irreducible tensor operators by G. Racah [Phys. Rev. (2) **62** (1942), 438–462]. — This book provides a self-contained and concise introduction to these methods. It gives a bibliography of recent applications and will be most valuable for physicists who want to apply these techniques to specific problems.

D. L. Falkoff (Waltham, Mass.)

2202:

Primas, H.; und Günthard, Hs. H. Eine Methode zur direkten Berechnung des Spektrums der von quantenmechanischen Systemen absorbierten bzw. emittierten elektromagnetischen Strahlung. Helv. Phys. Acta **31** (1958), 413–434.

The problem under consideration is the calculation of the frequencies ω_{jk} and intensities A_{jk} of the electromagnetic radiation emitted or absorbed by a quantum mechanical system. It is shown that if one considers an idealized spectrum $Q(\omega)$ of the form $Q(\omega)=\sum A_{jk}\delta(\omega-\omega_{jk})$, then the Fourier transform of $Q(\omega)$ has the properties of a correlation function and is given by

$$K(t)=\sum_{j<k} A_{jk}\cos\omega_{jk}t=\mathrm{Tr}\{X(t)X(0)\},$$

where $X(t)=\exp(-iHt)P\exp(iHt)$, H being the Hamiltonian of the system and P the interaction with the electromagnetic radiation. It is thus possible to get the spectrum without solving the eigenvalue problem. A certain generalization of this result is also discussed. Then it is shown that $K(t)$ is just the generating function for the moments used by Waller [Z. Physik, **79** (1932), 370–388], and by Van Vleck [Phys. Rev. **74** (1948), 1168–1183]. It is also possible to give an explicit expression of the intensities as functions of the frequencies and moments. Finally, a perturbation type approximation method is given which permits an explicit calculation of the correlation function. This method operates in the interaction representation in order to obtain a power series expansion in the "coupling constant" alone, without involving at the same time an expansion in t.

M. J. Moravcsik (Livermore, Calif.)

2203:

Regge, T. Analytic properties of the scattering matrix. Nuovo Cimento (10) **8** (1958), 671–679.

The Schrödinger equation $\psi''+(k^2-V(x))\psi=0$ has a solution $f(k,x)$ such that $e^{ikx}f(k,x)\to 1$ as $x\to\infty$. If $V(x)$ is sufficiently small at ∞, it is known that $f(k)=f(k,0)$ is entire; the existence and location of its zeros are matters of physical interest; it is known [H. Rollnik, Z. Physik **145** (1956), 654–661] that there are infinitely many zeros but only a finite number on the imaginary axis. The author supposes that $V(x)=0$ for $x>a$ and shows that even more can be said about the zeros of $f(k)$ by appealing to the general theory of entire functions, since $f(k)$ turns out to be of exponential type. For example, the zeros turn out to have equal density in the right-hand and left-hand half planes. A heuristic argument indicates that their distances from the real axis grow logarithmically; this depends on special information about the asymptotic behavior of $f(k)$ which is not available in the general theory. The author attempts to discuss the Hadamard product for $f(k)$ but arrives, without realizing it, at a divergent result; fortunately he makes no physical application of this.

R. P. Boas, Jr. (Evanston, Ill.)

2204:

Filippovič, E. I. Structure of divergent integrals of an S-matrix in α-representations. Ukrain. Mat. Ž. **10** (1958), no. 1, 84–89. (Russian)

2205:

Vladimirov, V. S. On a certain integro-differential equation. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 3–52. (Russian)

2206:

Vladimirov, V. S. On the integro-differential equation of particle transport. Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 681–710. (Russian)

This is a generalization of the results obtained by the author in an earlier paper [listed above] on the equation of particle transfer with isotropic scattering to the case of anisotropic scattering. The process of the transfer of particles is described by a linear integro-differential equation with certain boundary conditions. The object of the present work is the construction of a mathematical theory for the integro-differential equation mentioned above. The author makes use of the theory of Hilbert space and establishes theorems concerning the characteristic values and functions of the proposed problem.

H. P. Thielman (Ames, Iowa)

2207:

Sack, R. A. Taylor's theorem for shift operators. Phil. Mag. (8) **3** (1958), 497–503.

Specializing the following general formula of Kermack and McCrea:

$$F(q+f(p))=\exp(H)F(q),$$

where $H=f(p)d/dq+(\frac{1}{2}!)f'(p)(d/dq)^2+\cdots$ and $qp-pq=1$, to the case $f(p)=\lambda\exp(ap)$, the author obtains a Taylor theorem for the expansion of the function $F(A+\lambda B)$ in the non-commutative case $AB-BA=aB$. Application is made to the calculation of the matrix elements of $\exp(-aq^2)$ between the various eigenstates of the harmonic oscillator. *J. Schwartz* (New York, N.Y.)

2208:

Kolsrud, Marius. On adiabatic invariance in quantum mechanics. Norske Vid. Selsk. Forh., Trondheim **31** (1958), no. 5, 8 pp.

The author exhibits the connection between adiabatic invariance in classical and quantum mechanics for the special case of a particle moving in a homogeneous, time-dependent magnetic field.

D. Falkoff (Waltham, Mass.)

2209:

Yennie, D. R.; and Gartenhaus, S. Convergence of the *S*-matrix. Nuovo Cimento (10) **9** (1958), 59–76.

It is shown that the elements of the unrenormalized *S*-matrix, for linear interaction between a fermion and boson field which is restricted to a finite space-time region and has momentum cut-off, are entire functions of the interaction parameter. Hence, the elements of the renormalized matrix are quotients of entire functions in this case. From this result no certain conclusions were obtained as to the convergence of the asymptotic *S*-matrix for infinite time. In appendix I, an elementary proof is given of the following fact which also follows immediately from Hadamard's factorization theorem: an entire function of order one with no zeros is an exponential function.

A. J. Coleman (Toronto, Ont.)

2210:

Faĭn, V. M. Quantum phenomena in the radio spectrum. Uspehi Fiz. Nauk **64** (1958), 273–313. (Russian)

A review paper on the nature and magnitude of quantum effects in the radio-frequency region of the electromagnetic spectrum. Part I deals with the interaction of free electrons and cavity resonators. A major problem is the existence of incoherence effects which give rise to radio noise. Some critical comments are made on the supposition that observable quantum effects ascribable to the second quantization of the radiation field and to the wave nature of the electron are of importance. It is concluded that while such effects exist in principle they are not of great importance in a numerical sense.

Part II is concerned with quantum effects in radio spectroscopy of atomic and molecular systems. Natural line-widths and the coherent radiation from systems of molecules are the principal topics. Emphasis is placed on the numerical evaluation of quantities of physical interest. *E. L. Hill* (Minneapolis, Minn.)

2211:

Liotta, R. S. Some aspects of the covariant functional formalism of field theory. II. Nuovo Cimento (10) **8** (1958), 798–806.

[For part I, see Nuovo Cimento (10) **3** (1956), 438–446; MR **17**, 1165.]

The author describes a covariant formalism of quantum field theory, in which the usual Hamiltonian is replaced by an "invariant Hamiltonian", which is defined as the trace of the canonical energy-momentum tensor. The case of the electromagnetic field is briefly discussed.

S. N. Gupta (Detroit, Mich.)

2212:

Dyson, Freeman J. Connection between local commutativity and regularity of Wightman functions. Phys. Rev. (2) **110** (1958), 579–581.

Jost's version of the PCT theorem asserts that in relativistic quantum field theory the existence of PCT invariance is equivalent to a certain symmetry property of the vacuum expectation values of the theory called WLC by Dyson [R. Jost, Helv. Phys. Acta **30** (1957), 409–416; MR **19**, 712]. For the vacuum expectation value of the field operators $\psi_0(x_0)\cdots\psi_n(x_n)$, WLC is

$$\langle\psi_0(x_0)\cdots\psi_n(x_n)\rangle_0=\langle\psi_n(x_n)\cdots\psi_0(x_0)\rangle_0.$$

Furthermore, Jost showed that if this relation holds in the neighborhood of a single point $x_1, \cdots, x_n$ such that $\sum_{j=0}^{n}\lambda_j\xi_j$ is space-like for all λ_j, $j=1, \cdots, n$, satisfying $\lambda_j\geqq 0$, $j=1, \cdots, n$, $\sum\lambda_j>0$ (such a point will be called a Jost point hereafter), it holds everywhere. Here $\xi_j=x_{j-1}-x_j$, $j=1, \cdots, n$. Vacuum expectation values are analytic at all Jost points, because all Jost points are contained in the interior of the extended tube, R_n', which consists of all points $\Lambda\zeta_1, \cdots, \Lambda\zeta_n$ with Λ a complex Lorentz transformation, and $\zeta_1, \cdots, \zeta_n$ a point of the tube R_n defined by $\zeta_j=\xi_j-i\eta_j$, $j=1, \cdots, n$, η_j in the future light-cone, $j=1, \cdots, n$. It is known that all vacuum expectation values are boundary values of functions analytic in R_n'. (In particular, $\langle\psi_0(x_0)\cdots\psi_n(x_n)\rangle_0$ is the boundary value of an analytic function $W(\zeta_1\cdots\zeta_n)$ and $\langle\psi_n(-x_n)\cdots\psi_0(-x_0)\rangle_0$ that of an analytic function $V(\zeta_1\cdots\zeta_n)$.) However, not all points $x_1\cdots x_n$ such that x_j-x_k $(j\neq k=1, \cdots, n)$ are all space like (such points will be referred to as totally space-like) are Jost points. The paper under review establishes a connection between analyticity of vacuum expectation values at totally space-like points and WLC, in the following elegant theorem: Let S be the set of real points $\xi_1\cdots\xi_n$ at which $\langle\psi_0(x_0)\cdots\psi_n(x_n)\rangle_0$ has WLC. Then one of the following three alternatives holds. a) S is empty. b) S includes all Jost points. In this case the functions $W(\zeta)$ and $V(\zeta)$ are identical. They are single valued and analytic in a complex domain R including R_n'. A real point is in R if and only if it belongs to S. c) S is not empty and is disjoint from the set of Jost points. In this case W and V are two branches of a function analytic and single valued on a two-sheeted Riemann surface R covering R_n' twice. A real point belongs to S if and only if it lies within R at a place where the W branch over R_n crosses over into the V branch over $-R_n$. From this theorem two corollaries follow. 1) If PCT invariance holds, the set of real points at which $\langle\psi_0(x_0)\cdots\psi_n(x_n)\rangle_0$ has WLC is identical with the maximal set of real points in a complex domain R including R_n within which W is regular and single valued. 2) For the $\langle\psi_0(x_0)\cdots\psi_n(x_n)\rangle_0$ to have WLC at every point of a connected real domain D including all Jost points it is necessary and sufficient that PCT invariance hold and W be analytic in each point of D. The proof of the theorem is based on the "Edge of the Wedge" theorem of Bremermann, Oehme and Taylor [Phys. Rev. (2) **109** (1958), 2178–2190; MR **20**#3722] and a proof "making no pretensions to rigor" of that theorem is given. As it stands the latter proof is rigorous when the boundary values of W and V are continuous functions. It is so natural and simple that it is gratifying that Gårding and Beurling, in as yet unpublished work, have extended it to the case of boundary value distributions.

A. S. Wightman (Princeton, N.J.)

2213:

Corben, H. C. A new approach to the quantum theory of the electron. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 66–83.

The author starts from the standard anti-commutation relations of a Dirac field at a fixed time

$$[\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})^*]_+=\delta(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta},$$

$$[\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})]_+=0.$$

He defines for any 4×4 matrix of operators A, the operators $A^+=\int\psi^*(\mathbf{x})A\psi(\mathbf{x})d\mathbf{x}$ and $A^-=\int\psi(\mathbf{x})A\psi(\mathbf{x})^*d\mathbf{x}$, and derives a number of commutation relations for operators of this form. For example, if $p_k^+=-i\hbar\int\psi^*\partial_k\psi d\mathbf{x}$ and $p_k^-=i\hbar\int\psi\partial_k\psi^*d\mathbf{x}$, then

$$[x_k^+, p_l^+]=i\hbar[\delta_{kl}]^+,\ [x_k^-, p_l^-]=-i\hbar[\delta_{kl}]^-,$$

$$[x_k^+, p_l^-]=i\hbar[\delta_{kl}]^-.$$

He further asserts that $[\delta_{\alpha\beta}]^+ + [\delta_{\alpha\beta}]^-=0$, in contrast to usual theory, and claims "at least one divergence of quantized field theory — the vacuum charge expectation value is a consequence of the plane wave expansion used in the development of the theory." {In the reviewer's opinion, this last statement is false, and the author's procedure, in so far as it makes sense, coincides precisely with the subtraction introduced by Heisenberg [Z. Physik **90** (1934), 209–231]. For example, according to Heisenberg $[\delta_{kl}]^+$ is actually defined by

$$\int d\mathbf{x} \lim_{\mathbf{y}\to\mathbf{x}} [\psi^*(\mathbf{x})\psi(\mathbf{y})-\langle\psi^*(\mathbf{x})\psi(\mathbf{y})\rangle_0],$$

where the second term is a vacuum expectation value.} Now the idea is to interpret certain states of the electron positron field as particles with an internal structure. The states in question are constructed as proper functions of the operator $(\hbar/c)(\gamma_k\partial_k)^+$ and the charge for a free Fermi Dirac field, but the author suggests they might be reinterpretable as states of coupled electromagnetic and electron fields. The main formal content of this part of the paper is sets of equations which approximately describe the behaviour of the above mentioned states in terms of the time development of certain macroscopic observables. These latter are derived by setting up an analogy between the quantization of classical systems and second quantization. They are too complicated to describe in detail here. *A. S. Wightman* (Princeton, N.J.)

2214:

Frazer, W. R.; and Van Hove, L. The stationary states of interacting fields. Physica **24** (1958), 137–154.

The authors make some formal remarks concerning the connection between the physical and the bare particles in a quantized field theory. They use a modification of time independent perturbation theory developed earlier by Hugenholtz and Van Hove [Van Hove, Physica **21** (1955), 901–923; **22** (1956), 343–354; Hugenholtz, ibid. **23** (1957), 481–532; MR **19**, 610]. This method is not explicitly covariant and the ultraviolet infinities encountered in the discussion are treated with the aid of a non-covariant cut off. In this way the authors obtain explicit expressions for the expansion of the physical vacuum and the physical one particle states in terms of the bare particle states. The result is a formal power series in the interaction energy and the convergence of these expressions is not discussed. A similar formula for the S-matrix is also written down and all expressions are interpreted with the aid of diagrams. At the end of the paper a few formal remarks are also made about the renormalization constants. *G. Källén* (Lund)

2215:

Hurley, A. C. Role of atomic valence states in molecular energy calculations. J. Chem. Phys. **28** (1958), 532–542.

Ab initio calculations of molecular binding energies are examined critically in a basis of antisymmetrized products of approximate atomic state functions (the approximate composite functions of Moffitt). It is shown that much more accurate results may be expected if the binding energy is calculated relative to a suitably defined generalized valence state V of the dissociation products. The energy of this state V relative to the ground state of the dissociation products is estimated from atomic spectral data. This procedure leads directly to the intra-atomic correlation correction previously proposed by the author as a necessary modification of Moffitt's method of atoms in molecules. (From the author's abstract) *P.-O. Löwdin* (Uppsala)

2216:

Bodi, Lewis J.; and Curtiss, C. F. Theory of rotational-vibrational interaction in polyatomic molecules. J. Chem. Phys. **25** (1956), 1117–1122.

Group theoretical considerations show that an eigenfunction of the Schroedinger equation for a polyatomic molecule may be written as a sum of products of representative coefficients of the group of three-dimensional rotations and functions which depend only upon the relative configuration of the atoms and hence only upon the normal coordinates. A perturbation solution of the Schroedinger equation is described which is based on zero-order wave functions which are consistant with this group theoretical result. All of the angular momentum operators appearing in the wave equation are shown to be recursion operators in the representation coefficients. Subsequent utilization of the orthogonality of the representation coefficients then leads to a set of coupled differential equations for the functions which depend only on the normal coordinates. This set of coupled differential equations is a generalization of the matrix equations for the rotational motion of a rigid body.

The functions of the normal coordinates are then expanded as sums of products of Hermite functions of the individual normal coordinates. Use of the orthogonality relations of the Hermite functions then yields a set of equations for the expansion coefficients. The eigenvalues of the associated matrix are then, of course, the exact rotational-vibrational energy levels of the polyatomic molecule. These energy levels are obtained approximately by a diagonalization in the rotational quantum numbers followed by a perturbation technique in which the coupling between vibrational levels is assumed to be small (except in the case of degeneracy). The lowest approximation to the energy levels contain terms which may be interpreted as due to coupling effects which in previous solutions appear only in higher order approximations. (Author's summary) *A. C. Hurley* (Melbourne)

2217:

***Davison, B.; and Sykes, J. B. Neutron transport theory.** Oxford, at the Clarendon Press, 1957. xx+450 pp. 75 s.

The book provides the mathematical basis of the problem of the diffusion of neutrons as it occurs in reactor physics. The emphasis is on the exact solution whenever

that is possible; and on practical methods of approximate solution in other cases.

The book is in four parts. In the first part, the basic equations are formulated and general theorems concerning eigenvalues, eigenfunctions, their orthogonality and completeness are established. The next three parts are devoted to diffusion with constant cross-section, energy dependent diffusion with and without spectral regeneration and the asymptotic forms of the distribution at large distances.

The book is a substantial contribution to mathematical physics. *S. Chandrasekhar* (Williams Bay, Wis.)

2218:

Wanders, G. Limite non-relativiste d'une équation de Bethe-Salpeter. Helv. Phys. Acta **30** (1957), 417–446.

A few years ago, Wick and Cutkosky succeeded in solving the Bethe-Salpeter equation in a somewhat special case [G. C. Wick, Phys. Rev. **96** (1954), 1124–1134; R. E. Cutkosky, ibid., 1135–1141]. They considered scalar nucleons and scalar mesons in scalar interaction and investigated the special case of vanishing meson mass. As a result of their investigation it turned out that the eigenvalue spectrum had to be characterized by four quantum numbers instead of three as in the non-relativistic theory of two interacting particles. The paper reviewed here generalizes this result of Wick and Cutkosky to the case of a non-zero value of the meson mass. The author shows that the same four quantum numbers appear also in this case and that his solution in the limit of zero meson mass approaches the solution of Wick and Cutkosky. Further, the solution given here is shown to approach the wave function for a non relativistic Yukawa potential in the limit when the mass of the nucleon becomes very large. *G. Källén* (Lund)

RELATIVITY

See also 1807, 2015, 2193.

2219:

Fletcher, J. G. Dirac matrices in Riemannian space. Nuovo Cimento (10) **8** (1958), 451–458.

In adapting the Dirac equation to the demands of general relativity one is naturally led to consider the total derivative of a spinor field, (in n dimensions) $\psi_{|r}=\psi_{;r}-\Gamma_r\psi$, which is transformed by the same spinor transformation as ψ when a change of coordinate system is made. Here $\psi_{;r}$ is the ordinary covariant derivative and Γ_r is an $n\times n$ matrix.

For the Dirac γ matrices one has

$$\gamma_{j|r}=0=\gamma_{j;r}-[\Gamma_r, \gamma_j]$$

which implies $\xi\Gamma_r=\frac{1}{2}\gamma_{i;r}\gamma^i$ where ξ is the linear operator which acting on an $n\times n$ matrix M yields $\xi M=\frac{1}{2}(nM-\gamma_jM\gamma^j)$. In the present paper this equation is solved for Γ_r by finding a polynomial P_n in ξ which effectively represents $1/\xi$, i.e., P_n takes the same numerical values as $1/\xi$ when ξ is replaced by any integer between 1 and n. Γ_r is given explicitly for $n=1, \cdots, 5$. The method is elementary but tricky Clifford algebra.

A. S. Wightman (Princeton, N.J.)

2220:

***Kooy, J. M. J. On relativistic rocket mechanics.** Proceedings of the VIIIth International Astronautical Congress, Barcelona 1957, pp. 569–596. Springer-Verlag, Vienna, 1958. vii+607 pp. $29.75.

Most authors, treating rectilinear rocket flight in non gravitational space, consider the rocket plus exhausted masses as one mechanical system, and apply the relativistic law of conservation of momentum. In this article, only the rocket plus still included fuel is considered as the mechanical system and the differential equation of rectilinear motion, and also of curvilinear motion in non gravitational space, is directly derived from the basic equation of relativistic mechanics. Further the astronautical aspect of the relativistic time dilatation is discussed. The acceleration of the vehicle with respect to a general metric field is of paramount importance. In order to define this general metric field, the space time structure of the universe as a whole may play a fundamental part. *Author's summary*

2221:

Aitmurzaev, T. A method of solving unsteady gas flow equations with allowance for dissipative processes in the general theory of relativity. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 769–772. (Russian)

The author studies from the relativistic point of view the non-stationary flow of a viscous heat-conducting gas in its own field of gravity. The author assumes the metric of the field considered to be determined by

$$ds^2=-e^{\lambda}dr^2-r^2e^{\gamma}(d\theta^2+\sin^2\theta d\varphi^2)+e^{\nu}dt^2,$$

where λ, γ and ν are functions of r and t. The established set of five equations with five unknown functions ε, ρ, λ, γ and ν of r and t is solved by successive approximations so that the approximate values of the unknown functions are obtained. ε includes the internal heat energy and the residual energy of the particles of the gas themselves, while ρ denotes the density, i.e., the quantity of the residual mass in the unit of volume.

T. P. Anđelić (Belgrade)

2222:

Castoldi, Luigi. Attorno all'antica relatività di Weyl. Rend. Sem. Fac. Sci. Univ. Cagliari **27** (1957), 45–47.

The author remarks that a Weyl space admitting a parallel vector field is a Riemann space, and discusses the significance for Weyl's unified theory of some differences in detail which depend on whether the parallel field is covariant or contravariant.

F. A. E. Pirani (Chapel Hill, N.C.)

2223:

Onicescu, O. Introduction à une mécanique invariante des systèmes. Rev. Math. Pures Appl. **2** (1957), 167–180.

The paper begins with a semi-philosophical discussion of dynamical theories concluding, "Despite their mathematical interest, unified field theories do not seem to lead to physical results of any interest." A theory is then sketched, differing from General Relativity yet treating gravitation as an inertial effect. It is formulated in terms of the Poincaré-Cartan invariant: $p_i dq^i - H dt$, which is called the "inertial form". For a single particle the theory coincides with the dynamics of Special Theory of Relativity. For more than one particle, the inertial form contains interaction terms which could be interpreted as instantaneous Newtonian action at a distance. By an ad hoc choice of an arbitrary constant, fair agreement is obtained with the observed advance of the peri-

helion of Mercury and the deflection of light by the solar field. *A. J. Coleman* (Toronto, Ont.)

ASTRONOMY

See also 2112.

2224:

Kovalevsky, Jean. Sur la détermination des orbites elliptiques par la méthode de Laplace. Bull. Astr. **21** (1957), 161–193. (English, German and Russian summaries)

L'auteur développe une méthode pratique de détermination des orbites elliptiques d'après la méthode de Laplace qui se fait en deux temps. On commence par chercher une orbite approchée à l'aide des trois observations et ensuite on calcule une orbite définitive en se servant d'un nombre arbitraire d'observations. Soit α, α' et α'', δ, δ' et δ'' les conditions initiales des coordonnées équatoriales α, δ de la planète qui ont permis d'arriver à une certaine orbite et dont on connait les écarts d'avec la $n$ième observation.

Soit $\Delta\alpha$, $\Delta\alpha'$, $\Delta\alpha''$, $\Delta\delta$, $\Delta\delta'$, $\Delta\delta''$, les variations des données initiales. En introduisant une transformation des coordonnées appropriée on ramène la variation de toutes les variables du problème aux variations des données initiales $\Delta\alpha\cdots\Delta\delta''$. On obtient 4 systèmes d'équations qui admettent toujours une solution et une seule si $\cos\delta\neq 0$.

La méthode n'est plus applicable lorsqu'un certain déterminant est nul. Mais on peut tourner cette difficulté en remplaçant $\Delta\alpha''$ et $\Delta\delta''$ par deux autres variables appropriées.

Le seul cas d'impossibilité demeure lorsque $\cos\delta=0$, c'est-à-dire $\delta=\pm 90°$.

L'auteur applique sa méthode au calcul de la petite planète Brabantia (1342) à partir des six observations. Cette méthode est valable aussi dans le cas des inclinaisons faibles. *M. Kiveliovitch* (Paris)

2225:

Dibaĭ, È. A. The magnetogravitational instability of an infinite cylinder. Astr. Ž. **35** (1958), 253–256. (Russian. English summary)

The problem of the gravitational instability of an infinite gaseous cylinder in a homogeneous magnetic field parallel to its axis is considered in terms of the theory of small vibrations. Instability relative to longitudinal vibrations is investigated. It is shown that the presence of a magnetic field does not affect the instability, and the criterion of gravitational instability in this case coincides with the analogous criterion deduced when the magnetic field is absent.

As an example, a chain of globules formed as a result of the disintegration of a dark filament in Taurus is considered. The application of the results of the developed theory shows that the densities and masses of the globules are evidently sufficient for the formation, by gravitational condensation, of stars from these globules. (Author's summary.) *K. C. Westfold* (Pasadena, Calif.)

2226:

Safronov, V. S.; and Rouscol, E. L. On the hypothesis of turbulence in the protoplanetary cloud. Voprosy Kosmog. **5** (1957), 22–46. (Russian. English summary)

In the absence of external forces, Rayleigh's criterion for the stability of a non-uniform rotational flow is $\mu d\mu/dr>0$ where $\mu=mr^2\omega$ is the angular momentum at distance r from the axis. For a compressible gas this is generalized to

$$\mu\frac{d\mu}{dr}>\frac{r^3}{2\rho^2}\frac{dp}{dr}\left[\left(\frac{d\rho}{dr}\right)_{ad}-\left(\frac{d\rho}{dr}\right)\right],$$

where $(d\rho/dr)_{ad}$ is the adiabatic density gradient while $d\rho/dr$ and dp/dr are the prevailing gradients of density and pressure. Expressed in terms of temperature gradient, the condition is

$$(*)\qquad \frac{dT}{dr}>(\gamma-1)\frac{T}{\rho}\frac{d\rho}{dr}\left\{1+\frac{\gamma}{\gamma-1}\frac{\rho^2}{(dp/dr)(d\rho/dr)}\times\left[\frac{GM}{r^3}+\frac{3}{\rho r}\frac{dp}{dr}+\frac{d}{dr}\left(\frac{1}{\rho}\frac{d\rho}{dr}\right)\right]\right\},$$

where the gravitational influence of a central mass (M) determines the prevailing pressure gradient and γ is the ratio of the specific heats. A principal conclusion of the paper is that in view of (*), the generally assumed state of turbulence in an original solar nebula (from which the planets are supposed to have condensed) is invalid. The evolution of a system originally in a state of turbulence is also examined; and it is argued that the time of relaxation for the decay of the turbulence is too short for turbulence to be of any cosmological significance.

S. Chandrasekhar (Williams Bay, Wis.)

2227:

De, J. Stellar configuration with a toroidal magnetic field. Z. Astrophys. **44** (1958), 249–254.

By assuming that axisymmetric toroidal magnetic fields ($\mathbf{H}$) and fluid motions ($\mathbf{v}$) are present in an incompressible fluid, the author shows that if the ratio $|\mathbf{v}|/|\mathbf{H}|$ is a constant, we can obtain a simple special solution of the hydromagnetic equation. In case the motions correspond to a rigid rotation, the configuration is a sphere if the energy density in the magnetic field equals that in the velocity field; and it is an oblate or a prolate spheroid depending on whether the energy in the magnetic field is less than or greater than that in the motions [see also S. Chandrasekhar, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 273–276; Astrophys. J. **124** (1956), 571–579; MR **18**, 357, 969; S. K. Trehan, ibid. **127** (1958), 446–453; MR **20**#609; S. K. Trehan and W. H. Reid, ibid. 454–458; MR **20**#610]. *S. Chandrasekhar* (Williams Bay, Wis.)

2228:

Geršman, B. N.; and Ginzburg, V. L. Influence of magnetic field on convective instability in the atmospheres of stars and in the ionosphere of the earth. Trudy Gor'kov. Issled. Fiz.-Tehn. Inst. Radiofiz. Fak. GGU. Uč. Zap. **30** (1956), 3–29. (Russian)

The author points out that in a highly ionized gas, the electrical (σ) and the thermal (κ) conductivities are anisotropic when an external magnetic field is present. Thus the usual equation

$$\mathbf{j}=\sigma\left(\mathbf{E}+\frac{1}{c}\mathbf{u}\times\mathbf{H}\right),$$

relating the current ($\mathbf{j}$) to the electric ($\mathbf{E}$) and the magnetic ($\mathbf{H}$) fields and the motion ($\mathbf{u}$) via an electrical conductivity σ, becomes

$$(1)\qquad \mathbf{j}+\frac{\omega_e}{\nu_{ei}}(\mathbf{j}\times\mathbf{1}_H)=\sigma_{\parallel}\mathbf{E}+\frac{1}{c}\mathbf{u}\times\mathbf{H},$$

where ω_e ($=eH/mc$) is the gyration frequency of the

electron, ν_{ei} the frequency of electron-ion collisions and 1_H a unit vector in the direction of H. Similarly, the equation of heat conduction

$$\frac{\partial T}{\partial t}+(\mathbf{u}\cdot\operatorname{grad})T=\kappa\nabla^2 T$$

becomes

$$(2)\quad \frac{\partial T}{\partial t}+(\mathbf{u}\cdot\operatorname{grad})T=\left\{\kappa_{\|}\left(\sin\theta\frac{\partial}{\partial y}+\cos\theta\frac{\partial}{\partial z}\right)^2+\kappa_{\perp}\left[\frac{\partial^2}{\partial x^2}+\left(\cos\theta\frac{\partial}{\partial y}-\sin\theta\frac{\partial}{\partial z}\right)^2\right]\right\}T,$$

where $\mathbf{H}$ is assumed to act in the z-direction. Supplementing (1) and (2) by the remaining equations of motion and of Maxwell, the author investigates the problem of the inhibition of convection by a magnetic field. The analysis closely follows that of Chandrasekhar [Philos. Mag. **43**, (1952), 501–532; **45** (1954), 1177–1191; MR **14**, 813; **16**, 305] for the pure hydromagnetic case.

The modifications which arise when the gas is slightly ionized and the presence of neutral molecules is important are also discussed.

S. Chandrasekhar (Williams Bay, Wis.)

2229:

Geršman, B. N. Convective instability in a magnetoactive, completely ionized medium. Trudy Gor'kov. Issled. Fiz.-Tehn. Inst. Radiofiz. Fak. GGU. Uč. Zap. **30** (1956), 30–40. (Russian)

In the present paper the analysis of an earlier paper [see preceding review] is generalized to allow for the anisotropy of the viscous stresses in the presence of an impressed magnetic field. The expression for the stress given by Chapman and Cowling [The mathematical theory of non-uniform gases, Cambridge University Press, 1953; for review of 1939 edition, see MR **1**, 187] forms the basis of this discussion.

S. Chandrasekhar (Williams Bay, Wis.)

GEOPHYSICS

See also 2048.

2230:

Lineĭkin, P. S. On the influence of the surface of separation on wind waves on a deep sea. Trudy Gos. Okeanograf. Inst. **42** (1958), 89–104. (Russian)

2231:

Raichl, Jiří. Solution of a meteorological problem on punched-card machines. Stroje na Zpracování Informací **3** (1955), 101–127 (1956). (Czech. Russian and English summaries)

A meteorological problem, consisting of the initial tendencies of the 500 mb isobaric surface according to the barotropic vorticity equation, has been evaluated, using punched-card machines of the Powers type. The mathematical formulation of the problem is discussed, and the difference equations together with the boundary conditions are expressed. Methods of numerical solution of the difference analogue of the Poisson equation are considered. The most suitable method for punched-card machines seems to be that of the Fourier transformation, for which a detailed program is presented. It consists of a list of 37 manipulations of the punched cards. One computation involved about 300,000 arithmetic operations.

V. Vand (University Park, Pa.)

2232:

Vacca, Maria Teresa. Su alcuni moti, relativi allo spostamento del polo terrestre, dovuti a correnti cicliche marine. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 413–427.

Following a theory introduced by Volterra, the author discusses two cases of how the terrestrial poles may be displaced because of the angular momentum arising from the flow of water in the oceans. In both cases the total angular momentum R is the sum of the angular momentum K of the major part of the earth, conceived as a rigid body, and of the angular momentum W arisung from the ocean currents. In the first case W is perpendicular to R and rotates uniformly about it. In the second case the vector W is rigidly attached to the earth. The first case leads, under appropriate initial conditions, to the situation treated by Peano in which the north pole is displaced uniformly along a particular meridian (with respect to the initial pole) toward the initial equatioral plane. The second case leads to a system solvable by quadratures. If W lies in the equatorial plane, one meets significant cases solvable by elliptic functions.

D. C. Lewis, Jr. (Baltimore, Md.)

OPERATIONS RESEARCH AND ECONOMETRICS

See also 2071.

2233:

Bonferroni, Carlo. Les valeurs médianes et la théorie de la mesure. Publ. Inst. Statist. Univ. Paris **6** (1957), 53–67.

This is a translation of an article published in Giorn. Mat. Finanz. (4) **1** (1955), 89–110 [MR **19**, 372].

2234:

Bellman, Richard. Dynamic programming, successive approximations, and monotone convergence. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 578–580.

This short communication states that the application of dynamic programming and successive approximations enables one to approach the solution of a generalized Hitchcock-Koopmans transportation problem. The problem is given without details. An analogous result for another problem is retracted in an erratum.

L. Cesari (Baltimore, Md.)

2235:

Watanabe, Sigekatu. On the fundamental theorem for two-person rectangular games. Rep. Univ. Electro-Commun. no. **9** (1957), 11–23.

A rather complicated approach to the fundamental theorem in which one first obtains the Snow-Shapley formulas for extremal mixed strategies.

D. Gale (Santa Monica, Calif.)

2236:

***Holladay, John C. Cartesian products of termination games.** Contributions to the theory of games, vol. 3, pp. 189–200. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

Termination games are two-person games with perfect information in which the two players alternate and the

outcomes are win, lose, or draw only. The author further restricts his attention to win-or-lose games, and, by adding non-functional terminal moves, to terminal games in which the last player to play wins.

The author shows that for a finite termination game it is possible to associate with each position p a non-negative integer i so that i is the smallest integer for which there is no legal move into a position with the same value. The existence of such a function was first shown by P. M. Grundy [Eureka 2 (1939), 6–8] and it is now known as the G-value of such a game.

The author also considers direct products of such games and shows that the G-value of the composite game is the nim-sum of the G-values of the component games. This result has also been obtained by R. K. Guy and C. A. B. Smith [Proc. Cambridge Philos. Soc. 52 (1956), 514–526; MR 18, 546].

The latter, and greater, part of the paper deals with ordinal termination games in which the G-values are ordinal numbers. Results similar to but not identical with the results for finite termination games are obtained.

The author's assertion that a termination win-or-lose game can be considered as equivalent to a termination game in which the last player wins is misleading in the context of the rest of the paper. It is true that by adding non-functional moves at the end one can always obtain an equivalent game in which the person making the last move wins. However, when considering a cartesian product of games the last move in a component game is not the last move in the composite game. In such games the rules often demand that a move results in a real change in the position, and a non-functional move in a component game may do violence to the rules. That the situation is really much more complicated is shown by P. M. Grundy and C. A. B. Smith [ibid., 527–533; MR 18, 546].

E. D. Nering (Tucson, Ariz.)

2237:

***Dubins, L. E. A discrete evasion game.** Contributions to the theory of games, vol. 3, pp. 231–255. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

The game studied here is a zero-sum two-person game with imperfect information. One of the players, the evader, can turn either to the right or left at each moment of a discrete sequence of times. The other player, the pursuer, can attempt to bomb the evader at any one of these times. The pursuer knows where the evader is at the moment the bomb is released, but the bomb is delivered later, in this case two intervals of time later. Since it is assumed that a right-left is the same as a left-right, the pursuer knows that the evader will be in one of three positions when the bomb is delivered.

Dubins shows that the game has a value and that the evader has an optimal strategy, in fact a continuum of optimal strategies. However, there does not exist an optimal strategy for the pursuer.

E. D. Nering (Tucson, Ariz.)

2238:

Harling, John. Simulation techniques in operations research — a review. Operations Res. 6 (1958), 307–319.

Among the matters briefly reviewed are (i) generation of random or pseudo-random numbers, (ii) generation of random variates, (iii) techniques for the reduction of the variance of estimates, for a given amount of work, (iv) types of simulation [following the reviewer's classification in J. Roy. Statist. Soc. Ser. B, 16 (1954), 68–69], (v) selection of a model, (vi) an example of an industrial simulation. It is pointed out that a simulation undertaken by an industrial company often contains data that the company does not wish to disclose, and that this would explain the dearth of published case histories. (On page 311 the reference to item 2 in the bibliography should be to item 1.) *I. J. Good* (Cheltenham)

CONTROL SYSTEMS

2239:

Nishimura, Masataro. On the stability criterion of discontinuous servo-systems. Tech. Rep. Osaka Univ. 7 (1957), 253–262.

The nonlinear problem of the contactor control of a system with a second order transfer function (two real poles, one at zero) is discussed in detail. The relay has time delay and dead zone (hysteresis). The response of the system to an initial disturbance is studied by an exact method (analysis and phase plane) and the results are compared to those obtained by a linear analysis using the describing function of the nonlinear element (contactor). The describing function method gives good results as long as one does not go too close to the stability boundary. In the second part stability criteria for second order systems with backlash, deadzone or saturation are discussed. Only results obtained by the same methods are represented and compared to those obtained with help of the describing function.

I. Flügge-Lotz (Stanford, Calif.)

2240:

Krelmerman, M. M. Determination of parameters of corrective devices in linear servo-systems using given generalized parameters. Avtomat. i Telemeh. 19 (1958), 135–147. (Russian. English summary)

Analytical method of determining parameters of corrective devices in a linear servosystem using its generalized parameters is described. The paper includes the table of main formula and calculation of parameters of series and parallel corrective devices.

Author's summary

2241:

Segalin, V. G. Analytical formulation of the synthesis problem of corrective devices in linear servo-systems. Avtomat. i Telemeh. 19 (1958), 148–161. (1 insert) (Russian. English summary)

Analytical formulation of the synthesis problem of corrective devices in linear servo-systems is proposed. The determination of transfer functions of corrective devices is treated using initial equations of the synthesis obtained in paper. The method described is illustrated by an example of solving the synthesis problem.

Author's summary

2242:

***Barančuk, E. I. On the theory of multidimensional servo-systems.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 501–520. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

HISTORY AND BIOGRAPHY

2243:

***Smeltzer, Donald. Man and number.** Emerson Books, Inc., New York, 1958. viii+114 pp. $2.50.

This book gives a concise account of the use of number through the ages, from early man's acquisition of a sense of number and number words to the recording of numerical information and the carrying out of simple calculations in antiquity, medieval and modern times. It is shown how numbers were expressed in different civilizations, how the abacus was used, and how the Hindu system became our way of decimal reckoning. The last sections deal with division and sexagesimal and decimal fractions, and end with logarithms (a next edition might mention Chinese and Arabic use of decimal fractions). The author expresses special indebtment to the work of D. E. Smith. The bibliography has some interesting titles, though only English ones. The book, with its attractive illustrations, should have special fascination for pupils of elementary and secondary schools.

D. J. Struik (Cambridge, Mass.)

2244:

Frenkian, Aram M. Recherches de mathématiques suméro-akkadiennes, égyptiennes et grecques. I. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1(49)** (1957), 17–32.

The first paper of a series reviews various theories as to the origin of the base in the Babylonian sexagesimal system, and describes the classes of arithmetical tables found in the cuneiform sources. The standard {and, in the opinion of the reviewer, wrong} explanation for the origin of the Greek zero symbols as an initial omicron is here repeated. The author undertakes to show in a sequel that "Les Grecs... ne connurent sous aucune forme la numération de position...". {In view of the thousands of sexagesimal computations in the Greek sources, this should be a difficult thesis to sustain.}

E. S. Kennedy (Beirut)

2245:

Stamatis, Evangelos. Some remarks on successive approximations by iteration among the ancients. Prakt. Akad. Athēnōn **31** (1956), 336–343. (Greek. English summary)

2246:

Gericke, Helmuth. Geschichtliche Betrachtungen zur Situation der Mathematik. Math.-Phys. Semesterber. **6** (1958), 10–19.

A comparison of the circumstances under which the "Elements of geometry" of Euclid was written with the present circumstances under which the "Elements de mathématique" of Bourbaki is being produced.

2247:

Politano, Maria Luisa. Sull' "Analysis situs" di Leibniz. Archimede **9** (1957), 178–180.

2248:

Rychlík, Karel. Theorie der reellen Zahlen im Bolzano's handschriftlichen Nachlasse. Czechoslovak Math. J. **7(82)** (1957), 553–567. (Russian summary)

A part of the unpublished manuscript "Grössenlehre" by Bolzano (about 1830–34) is "Theorie der unendlichen Zahlen-(Grössen-)begriffe (ausdrücke)". It is an attempt at a systematic theory of real numbers and clarifies many concepts found in Bolzano's "Funktionenlehre" published only in 1930. The author gives those definitions and theorems of Bolzano contained in the manuscript, and by an interesting juxtaposition with the analogous ones of Cantor arrives at an historic evaluation of their merits and shortcomings in the analysis of the infinite.

S. R. Struik (Cambridge, Mass.)

2249:

Rychlík, Karel. Betrachtungen aus der Logik im Bolzano's handschriftlichen Nachlasse. Časopis Pěst. Mat. **83** (1958), 230–235. (Czech. Russian and German summaries)

An unpublished manuscript of Bernard Bolzano's "Grössenlehre" (National Library in Vienna) has a preface "Von der mathematischen Lehrart". The author singles out § 8 of it for publication because it contains problems of mathematical logic in the "Wissenschaftslehre" of Bolzano once more and perhaps still better formulated.

S. R. Struik (Cambridge, Mass.)

2250:

Bespamyatnih, N. D. On the history of calculating machines in Russia in the 19th century. Grodnen. Gos. Ped. Inst. Uč. Zap. Ser. Mat. **2** (1957), 43–54. (Russian)

2251:

Žautykov, O. A. Mathematics in Soviet Kazakhstan. Akad. Nauk Kazah. SSR. Trudy Sektor. Mat. Meh. **1** (1958), 5–24. (Russian)

An account of work done since its founding by members and adherents of the Kazakh Academy of Sciences. There is a bibliography with 114 entries.

2252:

Fichefet, J. Un mathematicien namurois meconnu François-Guillaume Poignard. Bull. Soc. Roy. Sci. Liège **26** (1957), 396–405.

2253:

Sobolev, S. L. On the 70th anniversary of Vladimir Ivanovič Smirnov. Izv. Akad. Nauk SSSR. Ser. Mat. 21 (1957), 449–456. (1 plate) (Russian)

A biography with a photograph and a bibliography of 109 entries.

2254:

Funk, Paul. Nachruf auf Prof. Johan Radon. Monatsh. Math. **62** (1958), 189–199.

A summary of Radon's scientific work, with one photograph and a bibliography of 45 entries.

2255:

Birkhoff, Garrett. Von Neumann and lattice theory. Bull. Amer. Math. Soc. **64** (1958), 50–56.

A review of von Neumann's various contributions to lattice theory, in particular to continuous geometries.

O. Ore (New Haven, Conn.)

2256:

Fukamiya, Masanori. Obituary note. Yoshitomo Okada. Tôhoku Math. J. (2) **10** (1958), 1–2.

2257:

Kamke, E.; und Zeller, K. Obituary: Konrad Knopp. Jber. Deutsch. Math. Verein. **60** (1957), Abt. 1, 43–49.

2258:

Anonymous. Biography: Enrico d'Ovidio. Archimede **9** (1957), 218–220. (1 plate)

2259:
Davies, E. T. Obituary: Frederick George Maunsell. J. London Math. Soc. **33** (1958), 255–256.

2260:
Todd, J. A. Obituary: Christine Mary Hamill. J. London Math. Soc. **32** (1957), 384.
Includes bibliography of three papers.

2261:
Davenport, H. Obituary: T. Vijayaraghavan. J. London Math. Soc. **33** (1958), 252–255.
The author describes this notice as based on the longer one by K. Chandrasekharan [Math. Student **24** (1956), 251–267; MR **19**, 108].

2262:
Szőkefalvi-Nagy, Béla. Riesz Frigyes 1880-1956. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **6** (1956), 144–156. (Hungarian)
An obituary, with a photograph. There is no bibliography.

2263:
Venkov, B. A.; and Šafarevič, I. R. Dmitriĭ Konstantinovič Faddeev (on the fiftieth anniversary of his birth). Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1(79), 233–238 (1 plate). (Russian)
A biography with a photograph and a bibliography of 38 entries.

2264:
Kolmogorov, A. N.; and Krasnosel'skiĭ, M. A. Mark Grigor'evič Kreĭn (on his fiftieth birthday). Uspehi Mat. Nauk (N.S.) **13** (1958), no. 3(81), 213–224. (1 plate) (Russian)
A biography with a photograph and a bibliography of 151 entries.

2265:
Anonymous. Artašes Liparitovič Šaginyan. (On the fiftieth anniversary of his birth.) Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **10** (1957), no. 1, 3–8. (Russian)
A biography with a photograph and a bibliography of 25 entries.

GENERAL

See also 2093.

2266:
***Gray, Dwight E. (Editor). American Institute of Physics Handbook.** McGraw-Hill Book Co., Inc., New York-Toronto-London, 1957. xi+1524 pp. $15.00.
This book encompasses the following sections: Mathematical aids to computation (editor: A. A. Bennett), mechanics (ed.: R. B. Lindsay), acoustics (ed.: F. A. Firestone), heat (ed.: M. W. Zemansky), electricity and magnetism (ed.: D. F. Bleil), optics (ed.: B. H. Billings), atomic and nuclear physics (ed.: G. H. Dieke), nuclear physics (ed.: F. N. D. Kurie).

2267:
Muller, H. J. Evolution by mutation. Bull. Amer. Math. Soc. **64** (1958), 137–160.
The thirty-first Josiah Willard Gibbs lecture (also Contribution no. 649 to the Department of Zoology, Indiana University), delivered at Cincinnati, Ohio, on January 28, 1958, under the auspices of the American Mathematical Society. The author expresses his confidence that evolution, in spite of his comparatively nonmathematical presentation of it, will be of great interest to mathematicians, as being its most exemplary products.

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